Knots and Singularities of Curves

Boston College Undergraduate Colloquium

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Knots

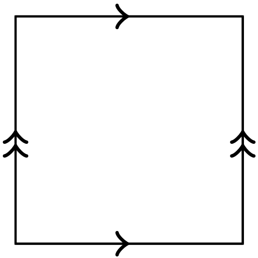
Definition

By a *knot* we mean a single string in 3 dimensions, possibly tangled around itself, which then has its ends joined together.

- We are allowed to move the string around in 3 dimensions, but we are not allowed to cut the rope or unjoin the ends.
- It can be *very* difficult to tell if two knots are the same.
- Knots are in our DNA: [VIDEO]

Example: Torus Knots

- A *torus* is the surface of a donut.
- Can visualize a torus by cutting and unfolding it into a rectangle, thinking of each axis as a circle.

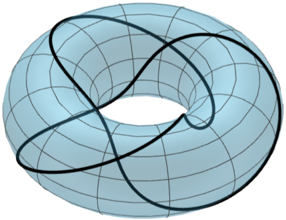


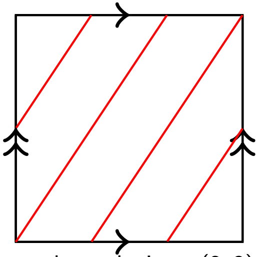
Example: Torus Knots

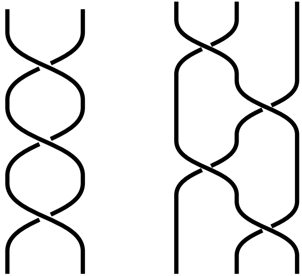
Definition

A *torus knot* is a knot that lives on the surface of the donut without crossings.

We measure a torus knot by two numbers: An (n, m) torus knot wraps n times round the cylinder of the donut and m times around the outer radius of the donut before it rejoins itself.







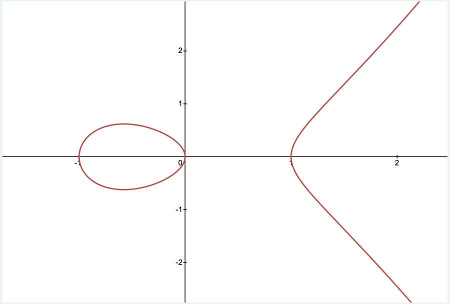
All the diagrams above depict a $(2, 3)$ torus knot.

Curves

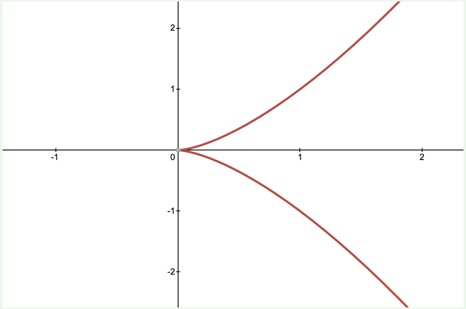
Definition

By a *curve*, we will mean the solutions of a single polynomial equation $p(x, y)$ in 2 variables. (Note: Our equations do not need to be functions!)

Examples



Graph of $y^2 = x^3 - x$. Here, $p(x, y) = y^2 - x^3 + x$.



Graph of $y^2 = x^3$. Here, $p(x, y) = y^2 - x^3$.

Singularities

Definition

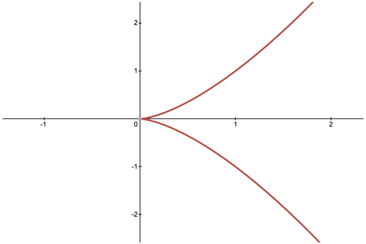
A *singularity* on a curve is a point where the curve is not differentiable; i.e. a point where the graph of the curve has a sharp turn.

You can detect singularities of an equation $p(x, y) = 0$ by solving for when all the partial derivatives are zero, i.e.

$$p_x(x, y) = p_y(x, y) = p(x, y) = 0$$

Example

The curve $y^2 = x^3$ has a singularity at $(0, 0)$.



Parametrizing Curves

Definition

- A *parametrization* of a curve is a set of directions for walking along that curve.
- The directions come in the form of a function $f(t)$, which outputs 2 dimensional coordinates for your position at time t.

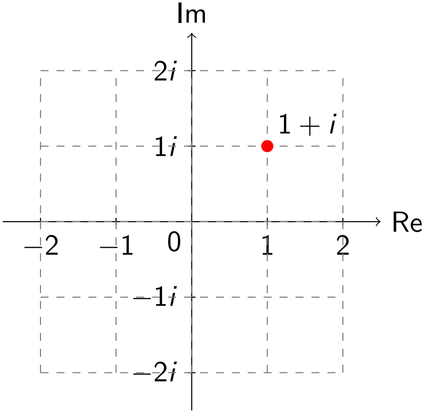
Examples

The curve $y^2 = x^3$ is parametrized by $f(t) = (t^2, t^3)$. [Animation]

The Complex Numbers

Definition

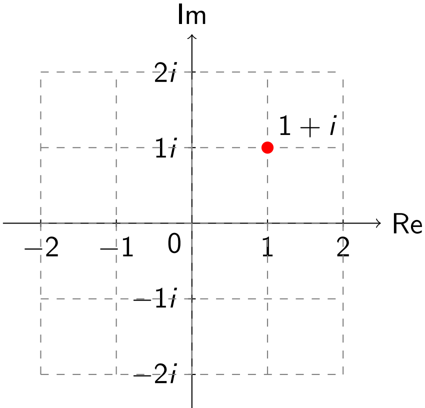
The complex numbers $\mathbb{C}$ are all the numbers of the form $a + bi$ where $i = \sqrt{-1}$.

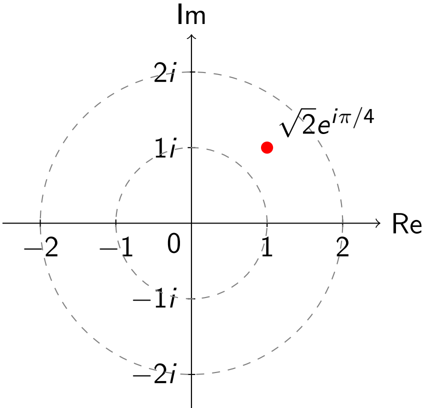


The Complex Numbers

- The space of complex numbers is 2-dimensional.
- Can always write $a + bi = re^{i\theta}$ using

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$





Curves over $\mathbb{C}$

Basic Motto

Instead of a 1-D curve in 2-D space, the solutions to $p(x, y) = 0$ in the complex numbers $\mathbb{C}$ form a 2-D surface in 4-D space.

Example

If the curve is $y = x$, then it looks like $\mathbb{C}$ itself. (Solutions to $y = x$ are of the form (u, u) in $\mathbb{C}^2$)

Example

If the curve is $y^2 = x^3 - x$, then it looks like a torus!

Question

What about the curve $y^2 = x^3$?

Parametrizing Curves over $\mathbb{C}$

- To parametrize a curve over $\mathbb{C}$, we have to replace the time t with a complex number $re^{i\theta}$.
- A parametrization is now a function $f(re^{i\theta})$, whose outputs are a pair of *complex* numbers (u, v) in $\mathbb{C}^2 = \mathbb{R}^4$.
- [ANIMATION OF SURFACE PARAMETRIZATION IN 3D]

Example

The curve $y^2 = x^3$ is parametrized by $f(re^{i\theta}) = (r^2 e^{2i\theta}, r^3 e^{3i\theta})$.

Question

How can we visualize this parametrization, which lives in 4 dimensions?

Visualizing in Dimensions You Can't Visualize

A Thought Experiment

Imagine trying to explain a curve to a 1-dimensional monster.

- This monster doesn't understand what 2 dimensional space looks like; you can't let teach this creature how to visualize it.
- How do you explain what the graph of the curve $y^2 = x^3$ "looks like" over $\mathbb{R}$ without referencing anything 2-dimensional?

One possible solution: Cut the two dimensional space up into one dimensional pieces and give the monster instructions how to glue it back together.

Examples

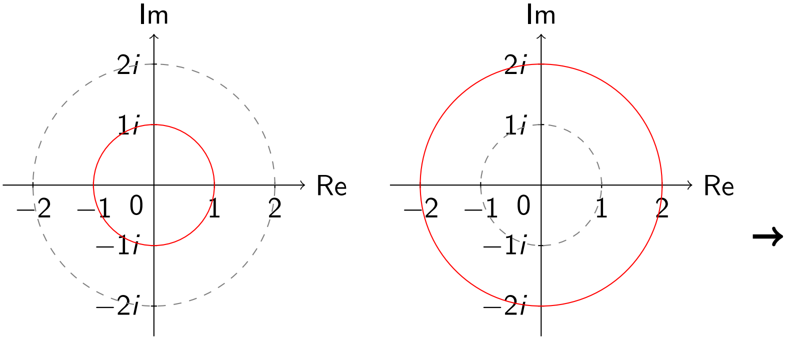
Consider $y^2 = x^3$ over $\mathbb{R}$. Cut 2-dimensional space up using concentric circles from the origin.

In each circle, there are two points, and we glue them all together as we shrink the radius.

[ANIMATION]

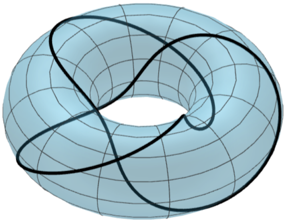
Becoming the Low-Dimensional Monster

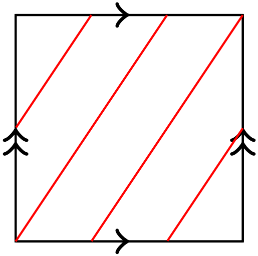
- Consider $y^2 = x^3$ inside of $\mathbb{C}^2$.
- Instead of using circles in $\mathbb{R}^2$, we can try to use *tori* in $\mathbb{C}^2$.
- This means fixing r and varying only θ in the parametrization $f(re^{i\theta}) = (r^2e^{2i\theta}, r^3e^{3i\theta})$.



The Curve $y^2 = x^3$

- We now want to see what $(r^2 e^{2i\theta}, r^3 e^{3i\theta})$ looks like when we fix r and let θ vary.
- [ANIMATION]





Conclusions

Summary

A higher dimensional friend may describe the graph of $y^2 = x^3$ as the "cone over the $(2,3)$-torus knot."
i.e. Take a line of (2,3) torus knots which are shrinking down to a point and glue them together to get a surface with a sharp point. [ANIMATION]

- In fact, in general, the curve $y^n = x^m$ is a cone over the (n, m) torus knot.
- This is the *beginning* of the story, not the end!
  - Have a knot attached to *any* curve singularity.
  - This knot is connected in deep ways to the geometry of the curve.
  - e.g. can recover polynomial knot invariants from interesting spaces ("Jacobians") coming from the singularity.
  - Knots form a bridge between different geometric objects coming from singularities. (e.g. Hilbert Schemes of Points, Affine Springer Fibers, etc.)

Thanks for listening!