

1. SYMPLECTIC REPRESENTATIONS AND DIVISORS

This note is to clarify a connection between divisors and symplectic representations that arose in [HLM25] as well as in forthcoming work of Ben-Zvi, Sakellaridis, and Venkatesh. Throughout, we let G be a reductive group and V a representation of G . In applications, we consider $G = G_X^\vee$ the dual group associated to a smooth, affine spherical variety with no type N roots and $V = S_X$ the symplectic representation of [BZSV24]. A central question this notes hopes to answer is to what degree a certain *branching divisor* actually determines the representation S_X in practice. Assuming some basic properties of the divisor $\mathfrak{D}$ and the representation S_X which are expected to hold in general, it turns out that the answer is that $\mathfrak{D}$ can completely determine the representation.

2. FROM REPRESENTATIONS TO DIVISORS

Let $\mathfrak{g} = \text{Lie}(G)$ and fix a maximal torus $T \subset G$ with associated Cartan $\mathfrak{t} = \text{Lie}(T)$. Denote by $\mathfrak{g} // G$ the GIT quotient

$$\mathfrak{g} // G := \text{Spec}(k[\mathfrak{g}]^G).$$

The famous Chevalley restriction theorem gives an isomorphism

$$\mathfrak{g} // G \simeq \mathfrak{t} // W,$$

where $W = N_G(T)/T$ is the Weyl group for G . To a representation V of G , we have a decomposition

$$V = \bigoplus_{\mu \in \mathcal{W}(V)} V_\mu$$

where $\mathcal{W}(V)$ is the space of weights $\mu: T \rightarrow \mathbb{G}_m$ such that the associated weight space

$$V_\mu = \{x \in V \mid t \cdot x = \mu(t)x\}$$

is nonempty. The function

$$\prod_{\mu \in \mathcal{W}(V)} \mu^{\dim V_\mu}$$

is a W invariant function on $\mathfrak{t}$ and so defines a divisor $\mathfrak{D}_V \subset \mathfrak{t} // W \simeq \mathfrak{g} // G$. Alternately, we can describe $\mathfrak{D}_V$ in the following way: The representation $\rho: G \rightarrow \text{GL}(V)$ induces a map on Lie algebras $d\rho$. Then, the conjugation invariant function $\det(d\rho)$ on $\mathfrak{g}$ defines $\mathfrak{D}_V$ on $\mathfrak{g} // G$.

3. RECOVERING THE REPRESENTATION V

We make the following assumptions on V , which are enjoyed by the symplectic representations S_X for *tempered* X in the relative Langlands program:

- (1) V is symplectic;
- (2) V has no zero weights (This holds conjecturally whenever X is smooth affine by [BZSV24, Remark 4.3.8(1)].);
- (3) The stabilizer of a generic element of V in G is connected.
- (4) V is coisotropic, i.e. the generic G orbit in V is coisotropic with respect to the symplectic structure on V .

Lemma 1. *Suppose V satisfies (1)-(4) above. Then all weights of V are primitive in the weight lattice.*

Proof. Recall $\mu \in \mathbb{X}^*(T)$ is primitive if and only if $\ker(\mu: T \rightarrow \mathbb{G}_m)$ is connected, and that this holds as soon as $\langle \mu, \alpha^\vee \rangle = \pm 1$ for some coroot $\alpha^\vee$; call such μ *coroot-primitive*. Note coroot-primitivity depends only on $(\Delta, \mathcal{W}(V))$, not on $\mathbb{X}^*(T)$.

By (1) and (4), V is a multiplicity free symplectic representation in the sense of [Kno06]. Run the algorithm of [Kno06, §3] on $(\Delta, \mathcal{W}(V))$, with output $(\Delta_0, \Phi_0 = \Phi_0^t \sqcup \Phi_0^s)$, $\Phi_0^t = \Phi_+^t \sqcup -\Phi_+^t$, and set $M = \{\lambda \in \mathbb{X}^*(T) : \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in \Delta_0\}$, a saturated sublattice of $\mathbb{X}^*(T)$.

Let $\mu \in \mathcal{W}(V)$. If μ is coroot-primitive we are done; by inspection of by inspection of Tables 1 and 2 of *loc cit*, the only weights that are not coroot-primitive are $\pm 3\varpi$ in $\langle 1.6 \rangle$, $\pm(2\varepsilon_i + \eta)$ in $\langle 2.3 \rangle$, and the

central weights $\pm\eta$ in (2.7) (m odd) and (2.9). The first two lie in Φ_0^t and the last two occur only in representations violating (3). So assume $\mu \in \Phi_0^t$. By (4) and [Kno06, Thm. 3.1], Φ_+^t is linearly independent. By [Kno06, Thm. 3.2] the generic stabilizer is $L_1 \times \prod_i \mathrm{Sp}_{2m_i-1}$ where $L_0 = \bigcap_{\chi \in \Phi_0^t} \ker \chi = L_1 \times \prod_i \mathrm{Sp}_{2m_i}$; as each Sp_{2m_i-1} is connected, (3) forces L_0 connected, and since $\pi_0(L_0) \cong (M/\langle \Phi_+^t \rangle)_{\mathrm{tors}}^\vee$ the sublattice $\langle \Phi_+^t \rangle$ is saturated in M . Being independent with saturated span, Φ_+^t extends to a $\mathbb{Z}$ -basis of M , so μ is primitive in M and hence in $\mathbb{X}^*(T)$. $\square$

Proposition 2. *Suppose that V enjoys properties (1)-(4) above. Then, we can recover V from the divisor $\mathfrak{D}_V \subset \mathfrak{g}/G$.*

Proof. The divisor $\mathfrak{D}_V$ tells us the weights of V up to scaling. Since V is symplectic with no zero weights, its weights come in pairs $\{\pm\mu\}$, and since all weights of V are primitive, this completely fixes the weights of V . The proposition follows as V is determined by its weights (they, for example, recover the formal character of V). $\square$

Remark 3. We note that (3) is essential for Lemma 1: For $G = \mathrm{SL}_2$ and $V = \mathrm{Sym}^3(\mathrm{std})$, conditions (1) and (2) hold but the generic stabilizer is $\mathbb{Z}/3$, and the weight 3ϖ is not primitive. Indeed, satisfying (1)-(3) is necessary for Proposition 2, as $V = \mathrm{Sym}^3(\mathrm{std})$ and $V = \mathrm{std}^{\otimes 2}$ share the same divisor $\mathfrak{D}_V$ for $G = \mathrm{SL}_2$.

We also note (4) is essential for Lemma 1: $\mathrm{Sym}^3(\mathrm{std}) \oplus \mathrm{std}$ satisfies (1)-(3) but is not coisotropic and has weight 3ϖ , which is not primitive.

4. WHICH DIVISORS COME FROM A REPRESENTATION?

Suppose now that we reverse the process and ask when a divisor $\mathfrak{D}$ arises from a unique representation V via the construction of Section 2. We focus on a special case of representations V built from minuscule weights here, which should conjecturally cover the representations attached to all affine, smooth spherical varieties X with no type N roots:

Definition 4. We denote by $\mathcal{C}$ the class of finite dimensional representations V of G satisfying the following:

- V satisfies conditions (1), (2), and (4) in Section 3;
- We may write $V \simeq \bigoplus_i V(\lambda_i)$ where $V(\lambda_i)$ is the highest weight representation of G of highest weight λ_i and all λ_i are minuscule and primitive.

Remark 5. We note that if $G = G_X^\vee$ and $V = S_X$ for a smooth, hyperspherical, affine spherical variety X with no type N roots, then V conjecturally lies in the class $\mathcal{C}$ by [BZSV24, Remark 4.3.8(1)] and [Bri, §5.1].

Let $\pi: \mathfrak{t} \rightarrow \mathfrak{t}/W \simeq \mathfrak{g}/G$ be the projection map.

Proposition 6. *Let $\mathfrak{D} \subset \mathfrak{g}/G$ be a divisor such that every irreducible component of $\pi^*\mathfrak{D}$ is a hyperplane $\mathfrak{h}_\mu$ for some minuscule $\mu \in X^*(T)$, and write $\pi^*D = \sum_{\mathfrak{h}} m_{\mathfrak{h}}[\mathfrak{h}]$. For each $\mathfrak{h}$ let $\pm\mu_{\mathfrak{h}}$ be the unique pair of primitive minuscule weights with $\mathfrak{h} = \mathfrak{h}_{\mu_{\mathfrak{h}}}$, and let $\lambda_{\mathfrak{h}}$ be the dominant representative of $W\mu_{\mathfrak{h}}$.*

- There is at most one V in the class $\mathcal{C}$ with $\mathfrak{D}_V = \mathfrak{D}$, namely $V = \bigoplus_{\mathfrak{h}} \frac{m_{\mathfrak{h}}}{2} V(\lambda_{\mathfrak{h}})$.*
- It exists if and only if $2 \mid m_{\mathfrak{h}}$ for all $\mathfrak{h}$, and $4 \mid m_{\mathfrak{h}}$ whenever $\lambda_{\mathfrak{h}}$ is self-dual of orthogonal type.*
- Such a V automatically satisfies conditions (1) and (2), and all its weights are primitive and minuscule.*
- Conditions (3) and (4) are additional restrictions on D . They hold if and only if, running the algorithm of [Kno06, §3] on $(\Delta, W(V))$, the set Φ_+^t is linearly independent and $\langle \Phi_+^t \rangle$ is saturated in $M = \{\lambda: \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in \Delta_0\}$. In particular $\deg \pi^*D = \dim V \leq \dim \mathfrak{g} + \mathrm{rk} \mathfrak{g}$ is necessary.*

Proof. If a V exists in $\mathcal{C}$ such that $\mathfrak{D}_V = \mathfrak{D}$, then an identical argument to the proof of Proposition 2 (together with the hypothesis that V is symplectic with primitive weights) implies that $V = \bigoplus_{\mathfrak{h}} \frac{m_{\mathfrak{h}}}{2} V(\lambda_{\mathfrak{h}})$ and that this expression is symplectic. This immediately implies that $m_{\mathfrak{h}}$ is even for all $\mathfrak{h}$. However, if $m_{\mathfrak{h}}/2$ is odd, then since V is symplectic, $V(\lambda_{\mathfrak{h}})$ must be symplectic. This is exactly saying that $\lambda_{\mathfrak{h}}$ is not self dual of orthogonal type. Hence, (a) and (b) hold. Parts (c) and (d) now follow by inspection on V . $\square$

We note that it remains a difficult problem to the author to *detect* from the divisor $\mathfrak{D}$ whether the associated representation V satisfies condition (3) of §3, which is saying that the spherical variety X is tempered.

REFERENCES

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