f_* & f^* are adjoint functors:

Prop. $f: X \to Y$. This is a canonical isom. (functorial in each input)

$$\mathcal{H}om(f^*\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om(\mathcal{F}, f_*\mathcal{G})$$

Rmk: In coherent category, this is locally:

$X \downarrow Y$, $\quad B \ni N^{\mathcal{G}}$, $\uparrow$, $A \ni M^{\mathcal{F}}$

$$\operatorname{Hom}_B(M \otimes_A B, N) \cong \operatorname{Hom}_A(M, N)$$

Special Case of:

$$\operatorname{Hom}_B(M \otimes_A P, N) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_B(P, N))$$

Pf: $\phi \in RHS$, $\mathcal{F} \to f_*\mathcal{G}$.

$\forall V \supset f(U)$, $f^{-1}(V) \supset U$

$$\Longrightarrow \quad \mathcal{F}(V) \longrightarrow f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}V) \xrightarrow{res} \mathcal{G}(U)$$

gives $\varinjlim_{V \supset f(U)} \mathcal{F}(V) \longrightarrow \mathcal{G}(U)$.

In reverse: $\psi: f^*\mathcal{F} \to \mathcal{G}$.

$\forall V \subset Y$ open, $f(f^{-1}V) \subset V$ gives

$$\mathcal{F}(V) \longrightarrow \varinjlim_{W \supset f(f^{-1}(V))} \mathcal{F}(W) = f^*\mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{G}(f^{-1}(V)) = f_*\mathcal{G}(V).$$

Can check these are inverse operations on stalks //

Consequences of an adjoint pair (f^*, f_*):

- f^* is right exact; f_* left exact [true for abelian cats.]
- More generally, left adj. preserves colimits: cokernels, direct limits, stalk; right adj. preserves limits: kernels, products, pullback, inverse lims, ...

OVER ⟶

6 functor formalism on $D^b_c(X)$: [a framework across many different situations!]

Def of $f_!$: For $s \in \Gamma(U, F)$, $\operatorname{supp}(s) = \{x \in X \mid s(x) \neq 0\}$.

e.g. If $U = \operatorname{Spec}(A)$ & $F = \widetilde{M}$ is coherent, $s \in M$ has

$$\operatorname{supp}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \neq 0 \text{ in } M_{\mathfrak{p}}\}$$
$$= \{\mathfrak{p} \mid \nexists a \in A \setminus \mathfrak{p} \text{ s.t. } a \cdot s = 0\} \cong \operatorname{Spec}\left(A/\operatorname{ann}_A(s)\right)$$
$$\rotatebox{90}{$\subseteq$}$$
$$\mathfrak{p} \supset \operatorname{ann}_A(s)$$

▷ $\Gamma_c(U, F) \subseteq \Gamma(U, F)$ is set of sections with compact support.

▷ Γ_c remains left exact & so has a right derived functor $R\Gamma_c$ [when $F = \underline{A}_X$, computes $H^i_c(X, A)$.]

$$R\Gamma_c \xrightarrow{\text{coh.}} H^i_c(X, F^\bullet) \text{ or } H^i_c(X, F).$$

▷ $f_! F$ is given by:

$$f_! F(U) = \left\{ s \in \Gamma(f^{-1}(U), F) \;\middle|\; f|_{\operatorname{supp}(s)} : \operatorname{supp}(s) \to U \text{ is proper} \right\}$$

supports are relatively cpt.

$f_! F \hookrightarrow f_* F$; isom. for proper F.

▷ $Rf_! F$ can be computed w/ "c-soft" shvs, i.e. those for which $\Gamma(X, F) \xrightarrow{\text{res}} \Gamma(K, F)$ is surjective $\forall$ compact $K \subset X$

[Here, $\Gamma(K, F) := \varinjlim_{U \supset K} \Gamma(U, F)$]

▷ $f_!$ & $Rf_!$ satisfy base change w/o properness assumptions!

Thm (Base Change).

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g'} & Y \end{array}$$

Cartesian. Then

▷ $(g')^* \circ f_! = f'_! \circ g^*$ in $\operatorname{Sh}(Y')$

▷ $(g')^* \circ Rf_! = Rf'_! \circ g^*$ in $D^b(Y')$

e.g. $j : U \hookrightarrow X$ open
$j_! F$ = extension by zero

/ **e.g.** $\mathbb{A}^2 - \{(0,0)\} \hookrightarrow \mathbb{A}^2$
$j_! \underline{\mathbb{Q}}$ = ext by zero; $(j_! \underline{\mathbb{Q}})_0 = 0$
$i_* \underline{\mathbb{Q}} = \underline{\mathbb{Q}}$; $(j_! \underline{\mathbb{Q}})_0 = \mathbb{Q}$

Def of $f^!$: —Very mysterious functor in general—

Hope: maybe $f_!$ has an adjoint functor $f^!$, i.e.,

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) \cong f_* \mathcal{H}om(\mathcal{F}, f^!\mathcal{G})$$

NOTE: $A=(A_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z} = \{\dots \cdot \cdot \cdot \dots\} \xrightarrow{f} \{pt\}$

$$f_* A = \prod_{n\in\mathbb{Z}} A_n \quad , \quad f_! A = \bigoplus_{n\in\mathbb{Z}} A_n$$

Having a left adjoint $\overset{\text{in abelian cat}}{\Longrightarrow}$ left exact

$\Longrightarrow$ preserves limits

$f_!$ cannot have left adjoint bc

$$\bigoplus_n \prod_k A \neq \prod_k \bigoplus_n A$$

→ Cannot exist in $Sh(X)$ also bc $f_!$ is not right exact.
But maybe it exists in $D^b_c(X)$:

① $j: U \hookrightarrow X$: open inclusion

$$\Gamma(U, j^!\mathcal{G}) = \mathrm{Hom}_U(\underline{A}_U, j^!\mathcal{G}) = \Gamma(U, \mathcal{H}om(\underline{A}_U, j^!\mathcal{G}))$$
$$= \Gamma(U, \mathcal{H}om(j_!\underline{A}_U, \mathcal{G})) = \mathrm{Hom}(j^*j_!\underline{A}_U, j^*\mathcal{G})$$
$$= \mathrm{Hom}(\underline{A}_U, j^*\mathcal{G}) = \Gamma(U, j^*\mathcal{G})$$

$\Longrightarrow j^! = j^*$ for open inclusions.

② $j: U \hookrightarrow X$; $f: X \to Y$

$$\Gamma(U, f^!\mathcal{G}) = \mathrm{Hom}_U(\underline{A}_U, j^*f^!\mathcal{G}) = \mathrm{Hom}_X(j_!\underline{A}_U, f^!\mathcal{G})$$
$$= \Gamma(Y, f_*\mathcal{H}om(j_!\underline{A}_U, f^!\mathcal{G}))$$
$$= \Gamma(Y, \mathcal{H}om(f_!j_!\underline{A}_U, \mathcal{G}))$$
$$= \mathrm{Hom}_Y(f_!j_!\underline{A}_U, \mathcal{G})$$

$\Rightarrow$ $f^!\mathcal{G}$ should be $[U \longmapsto \mathrm{Hom}(f_!j_!\underline{A}_U, \mathcal{G})]$.

NOT a sheaf! → replace $\underline{A}_X$ with a resol^n $\underline{A}_X \to K^\bullet$ [by c-soft sheaves]

$\Longrightarrow$ Can now define $f^!$ on derived category!

Method for $f^! \mathcal{G}$:

① $\mathcal{G} \longrightarrow \mathcal{I}^\bullet$ an injective resolution.

$\underline{A}_X \longrightarrow K^\bullet$: flat c-soft resⁿ.

If $\mathcal{I}$ is injective, $f^!_K \mathcal{I}(U) = \mathrm{Hom}_Y(f_! K^\bullet_U, \mathcal{I})$

$R : U \mapsto f_!(K^\bullet|_U)$ is a "cosheaf" valued in $Sh(Y)$

& $f^! \mathcal{G} := \mathrm{Hom}^\bullet_{D(X)}(R, \mathcal{G})$

Simple Cases:

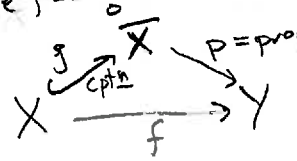
Ⓐ If $j: U \hookrightarrow X$ is open, then $j^! \cong j^*$ [exact]

Ⓑ If $j: Z \hookrightarrow X$ is c'sd, then $j^!$ is defined on sheaves &

$$j^! \mathcal{G}(U \cap Z) = \Gamma_{U \cap Z}(U, \mathcal{G})$$

(supp contained in $U \cap Z$)

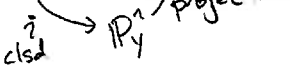
Ⓒ Better defⁿ:



$\Rightarrow f_! = p_* \circ j_!$

Prop. $f^! = j^* \circ p^!$ (reduces to p: proper.)

for projective $X \xrightarrow{f} Y$ also have $f^! = i^! \pi^!$



$\mathbb{P}^n_Y = Y \times \mathbb{P}^n$

▷ $i^!$ is done from before

▷ π is controlled by "dualizing ox": $\pi^! \mathcal{G} \cong \mathcal{G} \boxtimes \omega_{\mathbb{P}^n}$

$\omega_{\mathbb{P}^n} = \underline{A}_{\mathbb{P}^n}[2n]$

Ⓓ $f: X \to Y$ smooth:

$$f^! \mathcal{F} = f^* \mathcal{F} \otimes \omega_{X/Y}[d], \quad \omega_X = p_X^! \underline{A}_{pt}$$

constr. setting: $\omega_{X/Y} = or_{X/Y} = or_X \otimes_{\underline{A}} f^*(or_Y)^\vee$ (if X, Y are smooth) $= \underline{A}_X[2d]$, d = rel.dim

$\omega_X \otimes f^* \omega_Y^\vee$

Check for a local chart $U \cong \mathbb{R}^n \to \{pt\}$ using adjunction.

étale: $f^! \mathcal{F} = f^* \mathcal{F}(d)[2d]$.

[redacted]

Cor: $H^m(i_x^! \underline{\mathbb{Q}}_X) \cong \varinjlim_{U \ni x} H^m(U, U - x; \mathbb{Q})$

[vs. $H^m(i_x^* \underline{\mathbb{Q}}_X) \cong \begin{cases} \mathbb{Q}, & m=0 \\ 0, & m>0 \end{cases}$ always.]

Exact Δ: $U \xrightarrow{j} X \xleftarrow{i} X \setminus U$

$$Rj_! \circ j^!(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow Ri_* \circ i^*(\mathcal{F}) \xrightarrow{[1]}$$

(unit & counit of adjunctions)

why? On stalks: $(j_! j^! \mathcal{F})_x = (j_! j^* \mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$

$$(i_* i^* \mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in \partial X \setminus U \\ 0 & \text{if } x \in U \end{cases}.$$

Verdier Duality $\Longrightarrow$ Poincaré Duality for smooth X:

① Universal Coeffs: (A: Dedekind domain) $\forall$ open $U \subset X$, $\exists$ split ses

$$0 \to \mathrm{Ext}\left(\mathbb{H}_c^{q+1}(U, F^\bullet), A\right) \to \mathbb{H}^{-q}(U; \mathbb{D}_X F^\bullet) \to \mathrm{Hom}\left(\mathbb{H}_c^q(U, F^\bullet), A\right) \to 0$$

$f: X \to pt$, $Rf_* \mathbb{D}_X F^\bullet \cong \mathbb{D}_{pt}(Rf_! F^\bullet)$

Why? $R\Gamma(U, \mathbb{D}_X F^\bullet) \cong R\mathrm{Hom}_A(C^\bullet, A)$ for $C^\bullet = R\Gamma_c(U, F^\bullet)$

Then compute $R\mathrm{Hom}_A(C^\bullet, A)$ w/ spectral seq: A is dedek. dom. $\Rightarrow \exists\ A \to I^0 \to I^1$ inj. resol.

In particular, if A = field,

$$\mathbb{H}^{-q}(U, \mathbb{D}_X F^\bullet) \cong \mathrm{Hom}\left(\mathbb{H}_c^q(U, F^\bullet), A\right)$$

Prop. (Poincaré Duality). (A: field) X smooth of real dim n. Then

$$H^{n-i}(X, \mathcal{L}^\vee \otimes \mathcal{O}_X) \cong H_c^i(X, \mathcal{L})^\vee.$$

Pf: $\mathbb{D}_X \mathcal{L} \cong \mathcal{L}^\vee \otimes \omega_X \cong \mathcal{L}^\vee \otimes \mathcal{O}_X[n]$

Then $\mathbb{H}^{-i}(X, \mathbb{D}_X \mathcal{L}) \cong \mathbb{H}^{n-i}(X, \mathcal{L}^\vee \otimes \mathcal{O}_X) = H^{n-i}(X, \mathcal{L}^\vee)$

$\cong$ // univ. coeffs. $\mathrm{Hom}(H_c^i(X, \mathcal{L}), A) = H_c^i(X, \mathcal{L})^*$ (P.D.) //

Define: $f = i : Y \hookrightarrow X$ clsd inclusion of sm. mfds.

$$\mathbb{H}^*(X, X-Y; F^\bullet) := \mathbb{H}^*(X, Ri_! i^! F^\bullet)$$ [equal to usual notion for $F^\bullet = \underline{A}_X$]

Prop. (Alexander Duality): X oriented, $\dim X = n$, $i: Y \hookrightarrow X$ clsd, A = field.

$$\Longrightarrow \quad H^{n-i}(X, X-Y; A) \cong H_c^i(Y, A)^*.$$

Pf: $i_! i^! \mathbb{D}_X = i_! \mathbb{D}_Y i^* = \mathbb{D}_X i_! i^*$ ($i_! = i_*$). Apply to $\underline{A}_X = (\mathbb{D}_X \underline{A}_X)[-n]$

$$\Longrightarrow \quad i_! i^! \underline{A}_X[n] = \mathbb{D}_X i_! \underline{A}_Y. \quad \text{Apply } \mathbb{H}^{-k}(X, -).$$

$$H^{n-k}(X, X-Y; A) \cong \mathbb{H}^{n-k}(X; Ri_! i^! \underline{A}_X) = \mathbb{H}^{-k}(X, \mathbb{D}_X i_! \underline{A}_Y)$$

$$\cong H_c^k(X, i_! \underline{A}_Y)^* \cong H_c^k(Y, \underline{A}_Y)^* = H_c^k(Y, A)^* \quad //$$

(univ. coeffs)