Recap on abelian categories:

Working Def. An abelian category $\mathcal{A}$ is an additive category (i.e. s.t. $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is an abelian gp + mild conditions.) s.t.

$\forall\, A \xrightarrow{\phi} B$ in $\mathcal{A}$, $\exists\, \mathrm{Ker}(\phi), \mathrm{Coker}(\phi) \in \mathcal{A}$ s.t.

$\mathrm{Coim}(\phi) \longrightarrow \mathrm{Im}(\phi)$ is an isomorphism.

(quotient of A) (sub of B)

Defined via universal property: e.g. $\mathrm{Coker}(\phi) \in \mathcal{A}$ is

$A \xrightarrow{\phi} B \longrightarrow \mathrm{Coker}(\phi)$

$\forall\, f$ s.t. $f \circ \phi = 0$ $\quad \exists!$

Exs:
- Mod_A = cat. of A-modules
- $\mathrm{Sh}(X)$ = cat. of sheaves on X

This condition arises when one is not careful w/decorations one adds: e.g. {filtered vector spaces/k} is not abelian, but has Kernels & Cokernels.

Problem: $F^p k := \begin{cases} k & \text{for } p \geq 1 \\ 0 & \text{for } p < 1 \end{cases}$ & $\widetilde{F}^p k = \begin{cases} k & \text{for } p \geq 0 \\ 0 & \text{for } p < 0 \end{cases}$

$f : k \longrightarrow k$ is filtered, ie. $f(F^p k) \subseteq \widetilde{F}^p k$.

$\mathrm{coim}(f) = k$ w/ F-filtn

$\mathrm{im}(f) = k$ w/ $\widetilde{F}$-filtn

Sources of Constructible Sheaves: Cohomology of proper maps!

Next Goal: develop some homological alg. to define $R^i f_* \mathcal{L}$; $H^i(X, \mathcal{L})$.

Reminders on sheaf cohomology:

- Example of a derived functor: measures discrepancy of a functor from being exact.

Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be a functor (between abelian cats). F is right exact (resp. left exact) if whenever $0 \to A_1 \to A_2 \to A_3 \to 0$ is s.e.s., we have an exact seq

$F(A_1) \to F(A_2) \to F(A_3) \to 0$ (resp. $0 \to F(A_1) \to F(A_2) \to F(A_3)$)

Exs: f_* is left exact; $\Gamma(X, -)$ is left exact

$- \otimes F$ is right exact.

Thm/Def: $F: \mathcal{A} \to \mathcal{B}$ a left exact functor. Then its right derived functors $R^iF: \mathcal{A} \to \mathcal{B}$ are functors s.t.:

- ▷ R^iF is additive & $= 0$ for $i < 0$.
- ▷ $R^0F = F$
- ▷ Have a les $\forall$ses.
- ▷ functorial as map on ses
- ▷ R^iF is universal with these properties
- ▷ $\exists \forall A \in \mathcal{A}$, $A \hookrightarrow I$ s.t. $R^iF(I) = 0 \ \forall i > 0$

Some examples: ▷ $\Gamma(X, -)$ has right derived functor $H^i(X, -)$

▷ Derived functors of $f_*(-)$ gives $R^i f_*(-)$: sheaves whose fibers are coh. of fibers (proper base change!)

★ Thm/Def. does not show how to compute! ★

Practical Guide: To compute $F(A)$, replace A by a complex $A \to C^\bullet$.

For all of our purposes, we want A & $C^\bullet$ to be "the same"

▷ In particular, need $H^i(A) = H^i(C^\bullet)$, but this is not enough!

▷ Rather than give formulas, let's first explore what we mean by 2 complexes being "the same".

The Derived Category:

A: abelian category (e.g. $\mathrm{Loc}(X)$, $\mathrm{Sh}(X)$, etc.)

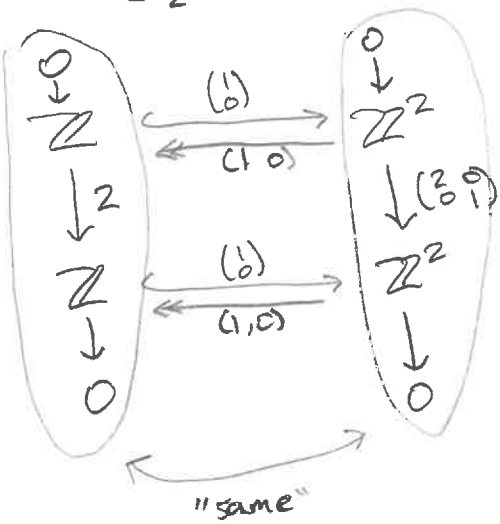
$C(A)$ = category of complexes

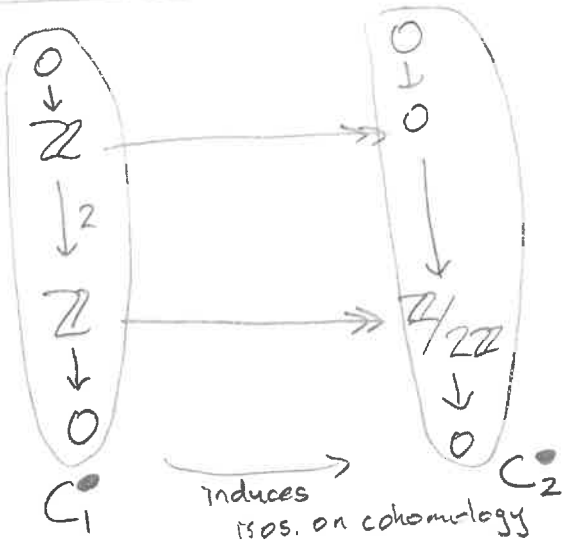
$$\cdots \to A_0 \to A_1 \to A_2 \to A_3 \to \cdots \quad \text{in } A$$

IDEA: want to treat 2 complexes as "the same" if $C_1^\bullet \to C_2^\bullet$ induces isomorphisms on (co)homology groups

Call these "quasi-isomorphisms"

Ex:





But: there's no inverse!

Working Def: $D(A)$ is a category whose objects are complexes & maps are usual maps + inverses of all quasi-isoms.

e.g. $C_1^\bullet \cong C_2^\bullet$ in $D(A)$.

Cost of this construction is that $D(A)$ is no longer abelian!

$D(A)$ is "triangulated": suppose $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ is a short exact sequence. For a complex $C^\bullet$, we have a "shift functor" $C^\bullet[1] = C^{\bullet+1}$

$$C^\bullet: \quad \cdots \to C^1 \to C^2 \to C^3 \to \cdots$$

$$C^\bullet[1]: \quad \cdots \to C^2 \to C^3 \to C^4 \to \cdots$$

$$\Longrightarrow \quad H^i(C^\bullet[1]) = H^{i+1}(C^\bullet)$$

Claim/Lem: There is a natural map $C^\bullet \to A^\bullet[1]$ in the derived category.

Pf: Put $\operatorname{cone}(f)^\bullet$:

$$\cdots \to \underset{B^n}{\overset{n}{A^{n+1} \oplus}} \xrightarrow{\partial} \underset{B^{n+1}}{\overset{n+1}{A^{n+2} \oplus}} \to \cdots$$

$$\partial(a,b) = (-\partial_A a, \partial_B b - f(a))$$

$$\partial = \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}$$

$$A^\bullet[1] \xleftarrow[\pi]{\text{proj}^n} \operatorname{cone}(f)^\bullet \xrightarrow{\psi} C^\bullet$$

$\psi(a,b) = g(b)$ [well-defined bc $\psi(\partial(a,b)) = g(\partial_B b - f(a)) = \partial_C g(b)$, with $g \circ f = 0$]

ψ is a qiso:

$$\operatorname{Ker}(\psi) = \{(a,b) \mid g(b) = 0\} = A^{n+1} \oplus f(A^n) \cong A^{n+1} \oplus A^n$$

$$\Rightarrow \operatorname{Cone}(1_A)^\bullet: \left[\cdots \to \underset{A^n}{\overset{n}{A^{n+1} \oplus}} \xrightarrow{\begin{pmatrix} -\partial_A & 0 \\ -1_A & \partial_A \end{pmatrix}} \underset{A^{n+1}}{\overset{n+1}{A^{n+2} \oplus}} \to \cdots \right]$$

- has trivial cohomology groups
- bc 1_A homotopic to 0: define $A^{n+1} \oplus A^n \xrightarrow{s} A^n \oplus A^{n-1}$

$$s(a,a') = (-a', 0)$$

Can check $ds + sd = 1_A$.

Reminder on homotopies:

Lem. If $f, g: A^\bullet \to B^\bullet$ are homotopic, i.e.

$$\cdots \to A^{n-1} \xrightarrow{\partial_A} A^n \xrightarrow{\partial_A} A^{n+1} \to \cdots$$
$$\cdots \to B^{n-1} \xrightarrow{\partial_B} B^n \xrightarrow{\partial_B} B^{n+1} \to \cdots$$

with f, g vertical and s the diagonal maps, s.t. $f - g = \partial_B s + s\partial_A$

Then f & g induce the same map on cohomology groups

Pf: $f(x) - g(x) = \partial_B(s(x)) + s(\partial_A x) = \partial_B(s(x))$ is a bdry

($x \in \ker(\partial_A)$) $\Rightarrow [f(x)] = [g(x)]$ in $H^n(B^\bullet)$ //

"Triangulated" Structure on $D^b(\mathcal{A})$:

has a collection of "distinguished triangles"

① Every map $A^\bullet \xrightarrow{f} B^\bullet$ fits into a unique distinguished triangle

$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{Cone}(f)^\bullet \xrightarrow{[1]} A^\bullet[1]$$

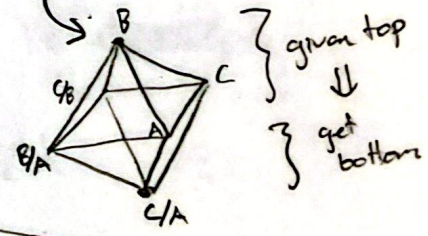
& $\mathrm{Cone}(1_A) \simeq 0$

② {distinguished triangles} is closed under $[1]$ & rotation $\left(A^\bullet \to B^\bullet \to C^\bullet \xrightarrow{[1]} A^\bullet[1] \rightsquigarrow B^\bullet \to C^\bullet \to A^\bullet[1] \xrightarrow{[1]} B^\bullet[1]\right)$

③
$$\begin{array}{ccccccc} A^\bullet & \to & B^\bullet & \to & C^\bullet & \xrightarrow{[1]} & \\ \downarrow & & \downarrow & & \exists!\downarrow & & \\ A'^\bullet & \to & B'^\bullet & \to & C'^\bullet & \xrightarrow{[1]} & \end{array}$$

Reflection of 3rd isom. thm: $\exists\, C/A$ fitting into $0 \to B/A \to C/A \to C/B \to 0$

④ [Technical] Octahedral Axiom: $\forall\ A \to B \to B/A$ & $\forall\ B \to C \to C/B$



Notes: Hom spaces between cxs. remember

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{p \in \mathbb{Z}} \mathrm{Hom}(A^p, B^{p+n})$$

$\partial^n = \prod_p \partial_p^n$ & for $\partial_p^n : \mathrm{Hom}(A^p, B^{p+n}) \to \mathrm{Hom}(A^p, B^{p+n+1})$

by $\partial_p^n(f_p) = \partial_{B^\bullet}^{p+n} \circ f_p + (-1)^{n+1} f_{p+1} \circ \partial_{A^\bullet}^p$

Lem: $\Gamma(X, \mathrm{Hom}^\bullet(F^\bullet, G^\bullet)) = [F^\bullet, G^\bullet] = \{\text{homotopies } F^\bullet \to G^\bullet\}$

(see Back!)

Pf that $H^0\Gamma(X, \mathcal{H}om^\bullet(A^\bullet, B^\bullet)) = [A^\bullet, B^\bullet]$:

$$\Gamma(X, \mathcal{H}om^n(A^\bullet, B^\bullet)) = \prod_p \mathrm{Hom}_{\mathcal{O}_X}(A^p, B^{p+n})$$

$$\mathrm{Ker}(\partial^0) = \left\{ (f_p) \in \prod_p \mathrm{Hom}_{\mathcal{O}_X}(A^p, B^p) \;\middle|\; \overbrace{\partial_B \circ f - (-1)^n f \circ \partial_A}^{\partial(f_p)} = 0 \right\}$$

↑ chain map cond^n

$$\mathrm{Im}(\partial^{-1}) = \left\{ \partial_B^{p-1} \circ h^p + h^{p+1} \circ \partial_A^p \right\} = \{\text{homotopies}\}$$

$h_p : A^p \to B^{p-1}$

//

W

Pf. octahedral identity holds?

- Back to derived functors: -

Recall that we want to build "derived functors" $Rf_*, H^*, -\overset{L}{\otimes}F, -$

By: ① Replace F w/ a "good" complex $C^\bullet$ qiso to F.
② Compute $F(C^\bullet)$ & take cohomology.

Idea: replace F w/ qiso. complex $C^\bullet$ s.t. $R^iF(C^a) = 0 \ \forall i > 0$.
→ this depends on F!

ie. To compute $F(\mathcal{F})$ suffices to take a resolution of $\mathcal{F}$ by objects that are F-acyclic, e.g. C is F-acyclic iff any surj^n $B \twoheadrightarrow C \implies F(B) \twoheadrightarrow F(C)$.

Most common "F-acyclics" are projective & injective sheaves:

Projective:

$$\begin{array}{ccccc} & & P & & \\ & \exists\swarrow & \downarrow & & \\ \forall\ A & \twoheadrightarrow & B & \to & 0 \end{array}$$

Injective:

$$\begin{array}{ccccc} 0 & \to & A & \hookrightarrow & B \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

Warning: ▷ $Sh(X)$ has enough injectives, but not enough projectives

▷ Injectives are awful: they never compute anything...

→ $\mathcal{F} \hookrightarrow \prod_{x \in X} i_{x,*} I_x$ for I_x : injective module w/ $I_x \twoheadrightarrow \mathcal{F}_x$.

Injective modules: ▷ over $\mathbb{Z}$, must be divisible: $\mathbb{Q}$ (true for any PID)

▷ If R is a domain & not a field, every injective R-module is ∞-dim'l.

Warning: "Projective" sheaf $\not\Rightarrow$ locally free !
(locally a projective module)

Why? $\mathcal{F} = \mathcal{O}$,

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O} \to \mathcal{O}$$

Γ is not exact on this:

les: $0 \to 0 \to 0 \to k \to k^2 \to k \to 0$

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What sheaves are acyclic?

▷ Injectives are acyclic $\forall$ left exact F.
↳ standard def^n/construction uses these.

▷ Flasque/flabby: when restriction maps are always surjective

↳ Prop. $\mathcal{F}$ flasque $\Rightarrow$ $\mathcal{F}$ is Γ & f_* acyclic.

Pf: $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ s.e.s. Then $\mathcal{G} \to \mathcal{H}$ is locally surjective $\Rightarrow \exists\, g_i \in \Gamma(U_i, \mathcal{G}) \mapsto h|_{U_i}$

on $U_i \cap U_j$, $\beta(g_i - g_j) = \beta(g_i) - \beta(g_j) = h|_{U_i \cap U_j} - h|_{U_i \cap U_j} = 0 \Rightarrow \exists\, f_{ij} \in \Gamma(U_i \cap U_j, \mathcal{F})$ s.t. $f_{ij} = g_i - g_j$

Extend $f_{ij} \in \Gamma(U_j, \mathcal{F})$ lift & $g_j \mapsto g_j + \tilde{f}_{ij}$.

Now $g_i = g_j$ on $U_i \cap U_j$. Then continue... $\Rightarrow H^1(X, \mathcal{F}) = 0$

Then use LES for $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{I}/\mathcal{F} \to 0$

▷ For left derived functors, use flat sheaves $\mathcal{B}^\bullet \longrightarrow \mathcal{F}$: qfso.

NOTE: It is still very hard to compute $f_*\mathcal{F}$ in general! [will eventually do so w/ dec. thm!]

General Philosophy: Better to work w/ complexes than cohomology.

▷ Consider $R^i f_* \mathcal{L}$ in more detail: What is this sheaf roughly? ($\mathcal{L}$ = local system)

other page ① Proper base change: $(R^i f_* \mathcal{L})_y = H^i(X_y, \mathcal{L}|_{X_y})$ ↳ local system on fiber.

② If $\mathcal{L} = \underline{\mathbb{Q}}_X$, Then $H^i(X, \underline{\mathbb{Q}}_X) \cong H^i(X, \mathbb{Q})$

Pf: Let $C^n(X, \mathbb{Q}) =: C^n(X)$ denote singular co-chains & $C^n : [U \mapsto C^n(U)]^\#$. then $\underline{\mathbb{Q}}_X \longrightarrow C^\bullet$ is a resolution. (uses local contractability)

C^n is flabby: can always extend a cochain on U by 0.

$\Gamma(X, C^n) = C^n(X)$, so this proves it.

③ If not, $H^i(X, \mathcal{L}) =$ cohomology w/ $\mathcal{L}$-coeffs [give formulat example]
other page

Rmk: $R^i f_* \mathcal{L}$ is constructible for $\mathcal{L}$: constructible on X & $f: X \to Y$ proper. (Uses same proof as before!)

$\rightsquigarrow$ **Def.** $\star$ $D_c(X)$ [or $D_c^b(X)$] denotes the full sub-cat. of $D(X) = D(Sh(X))$ [or $D_c^b(X) = D^b(Sh(X))$] of $C^\bullet$ s.t. $H^i(C^\bullet)$ is constructible $\forall i$.

(assume also all local systems involved are finite rank!)

$\star$ $Constr(X) :=$ category of constructible shvs on X.

Prop: $D^b(X) \simeq D^b(Constr(X))$

Pf Idea: for $\mathcal{L} \in Constr(X)$ & $\mathcal{F} \in Sh(X)$, s.t. $\mathcal{F} \to \mathcal{L}$, $\exists\, \mathcal{L}' \to \mathcal{F}$ s.t. $\mathcal{L}' \to \mathcal{F} \to \mathcal{L}$ is surj^n. & $\mathcal{L}' \in Constr(X)$

Build $\mathcal{L}'$ by lifting local gens of $\mathcal{L}$ to $\mathcal{F}$

Philosophy: complexes are better than cohomology.

Exs: ① $R(F \circ G) \cong RF \circ RG$ for left exact F, G

[vs. spectral sequence $R^pF(R^qG) \Rightarrow R^{p+q}(F \circ G)$]

② Universal coeffs: $C^\bullet$ = cx computing $H^i(X, \mathbb{Z})$.

cx computing $H^i(X; A)$ is $C^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} A$.

(vs. $H^i(X; A) \cong (H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} A) \oplus Tor^1(H^{i+1}(X, \mathbb{Z}), A)$)

Consequence: Often later try to work directly in $D_c^b(X)$.

Next Goals: ▷ 6 functor formalism in $D_c^b(X)$ [& introduction to $f_!, f^!$.

▷ Verdier duality in $D_c^b(X)$

$\rightsquigarrow$ opens up calculus of the derived category!