(15) # Crash Course on Sheaf Theory:

New Goal: Sheafify everything we talked about.

$IH_i \rightsquigarrow IC$ sheaves

We want this for a few reasons:

- ▷ Can start to deal with families.
- ▷ Can understand relationships between IC sheaves on subvarieties of ambient spaces / sources & targets of maps
- ▷ Categorifies constructions in geometric rep. theory, number theory / Langlands program, etc.

Def. A presheaf is a contravariant functor

$$\mathcal{F}: \{\text{open sets in } X\} \longrightarrow \{A\text{-modules}\}$$

(e.g. abelian gps.; $\mathcal{O}_X$-modules; vector spaces; ...)

$$\mathcal{F}(U) =: H^0(U, \mathcal{F}) =: \Gamma(U, \mathcal{F})$$

$\mathcal{F}$ is a sheaf if it also satisfies the gluing property: • If $s_i \in \Gamma(V_i, \mathcal{F})$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ $\forall i,j$

Then $\exists!\ s \in \Gamma(\bigcup_i V_i, \mathcal{F})$ s.t. $s|_{V_i} = s_i$ $\forall i$

[Note: might be infinitely many V_i !]

e.g. ① $M \in A\text{-mod}$.

$\underline{M}_X$ = constant sheaf on X

$\underline{M}_X(U) := \{\text{cts. } f: U \to M\} \cong M^{\#\pi_0(U)}$

② "Structure sheaf" of X, $\mathcal{O}_X = \{\text{sheaf of fcns on } X\}$

If X is a scheme & $U = \mathrm{Spec}(R) \subset X$ is an affine patch, $\mathcal{O}_X(U) = R$.

Def. The stalk of $\mathcal{F}$ @ $x \in X$ is $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$

(16) There are 2 flavors of sheaves that will be interested in: coherent & constructible

Coherent: Any algebraic variety, X, comes with a structure sheaf $\mathcal{O}_X$ of fcns on X.

$$\mathcal{O}_X(U) = \{\text{algebraic fcns on } X\}$$

A coherent sheaf is a sheaf $\mathcal{F}$ on X s.t.

▷ $\mathcal{F}(U)$ is a fin. presented $\mathcal{O}_X(U)$-module for all small enough U

If $U = \mathrm{Spec}(A) \subseteq X$ is an affine (Zariski) open set, $\mathcal{O}_X(U) = A$ & $\mathcal{F}|_U = \tilde{M}$ for a f.g. A-module M (i.e. $\mathcal{F}(U - (a)) = M_{(a)}$.)

NOTE: "Coherent" usually means Zariski topology!

Constructible:

- Def. An A-local system on X is a locally constant sheaf $\mathcal{L}$ of A-mods. i.e. $\exists$ open covering $X = \bigcup_i U_i$ s.t. $\mathcal{L}|_{U_i} \cong M_i$ for an A-mod M_i.

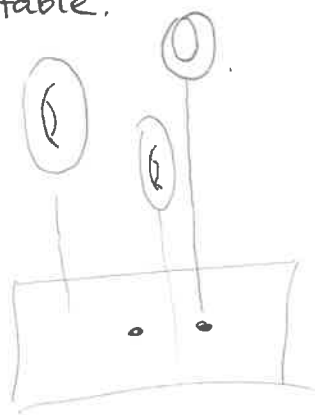
▷ If X is connected $M_i = M$ $\forall i$ is constant.

▷ If $M = M_i = A^r$ $\forall i$, then $\mathcal{L}$ is of "rank r".

Exs: ① (from topology) $\mathcal{O}_X^{naive}(U) = H_n(X, X-U; A)$

$\mathcal{O}_X = (\mathcal{O}_X^{naive})^{sh}$

$\mathcal{O}_X$ is constant $\iff$ X is orientable.



② (family) $\mathcal{X} = \{y^2 = x(x-1)(x-\lambda)\}$

$f \downarrow$

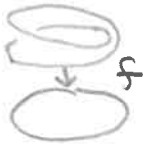
$\mathbb{A}^1_\lambda \setminus \{0,1\}$

$\mathcal{F}(U) = H_1(f^{-1}(U), \mathbb{Z})$ } stalks are $\mathbb{Z}^2$

(17) How to tell what a locally constant sheaf is?
(Or even whether it is nontrivial??)

A: Walk along paths in X & see if you end up in the same place!

Ex: Consider double cover of S^1



$\mathcal{F}(U) := f_* \underline{A}_{S^1}(U) := \underline{A}_{S^1}(f^{-1}(U))$
is locally constant. $\mathcal{F}_x \cong A^2$ (rank 2 local system)

Can trivialize $S^1 = U_1 \cup U_2$
$\mathcal{F}|_{U_i} \cong \underline{A^2}_{U_i}$

$(a_1, a_2) \in \mathcal{F}(U_1) \longrightarrow (a_1, a_2) \in \mathcal{F}(U_2) \longrightarrow (a_2, a_1)$

$\longrightarrow \pi_1(S_1, x_0) \longrightarrow \mathrm{Aut}(A^2)$
$1 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ order 2 image

Prop. X: connected top. space. Then TFAE:
(a) A-local systems on X w/ fibers M
(b) representations $\rho: \pi_1(X, x_0) \to \mathrm{Aut}_A(M)$.

Pf: ① $\mathcal{L}$: local system $\Rightarrow$ $\rho_{\mathcal{L}}: \pi_1(X, x_0) \longrightarrow \mathrm{Aut}_A(M)$

Bc: $p: [0,1] \longrightarrow X$ gives a trivial local system $p^{-1}(\mathcal{L})$ on $[0,1]$ & $p^{-1}\mathcal{L}_0 \cong \mathcal{L}_{p(0)}$, $p^{-1}\mathcal{L}_1 \cong \mathcal{L}_{p(1)}$.

② $\rho: \pi_1(X, x_0) \to \mathrm{Aut}_A(M)$ $\Rightarrow$ $\mathcal{L}_\rho$: local system

Bc: $\mathcal{L}$ = sheaf of sections of $(\tilde{X} \times M)/\pi_1(X, x_0)$
for $\tilde{X} \to X$: universal cover.

$\rho_{\mathcal{L}_\rho} = \rho$: Sheaf of sections of $(\tilde{X} \times M)/\pi_1(X, x_0)$ has monodromy:

$\gamma \in \pi_1(X, x_0)$ acts by $T_\gamma(s(\tilde{x}_0)) = s(\tilde{\gamma}(1)) = s(g_\gamma \tilde{x}_0) = \rho(\gamma)\, s(\tilde{x}_0)$
$s: p^{-1}(U) \to M$ loc. const. s.t. $s(g \cdot \tilde{x}) = \rho(g) \cdot s(\tilde{x})$

$\mathcal{L}_{\rho_{\mathcal{L}}} = \mathcal{L}$: $\tau: M_{\tilde{x}} \xrightarrow{\sim} p^*\mathcal{L}$ $\circlearrowleft$ Deck gp. π_1, $\tau_{g\gamma\tilde{x}} = T_\alpha \circ T_\gamma = \tau_{\tilde{x}} \circ \rho(\gamma)$

(Harder example) Recall the elliptic curve example.

Consider $\mathcal{L}(u) = H_1(f^{-1}(u))$. ← fibers $\mathbb{Z}^2$ $\pi_1(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \cong F_2$.

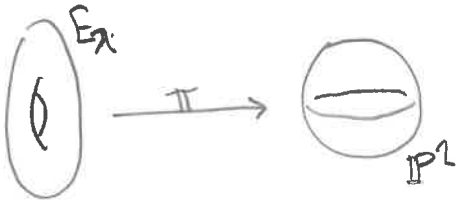
Q: What is $\rho: F_2 \longrightarrow \mathrm{Aut}(\mathbb{Z}^2)$ corresponding to $\mathcal{L}$?

A: $\rho(e_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ "Dehn twist"



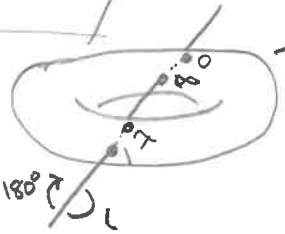
(more in Geordie's notes...)

Pf:





Topologically: π:



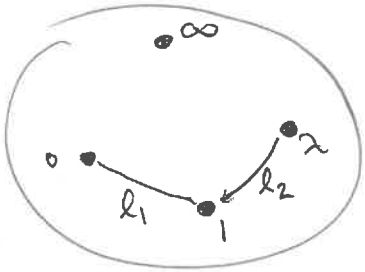
$\longrightarrow T/\iota \cong$

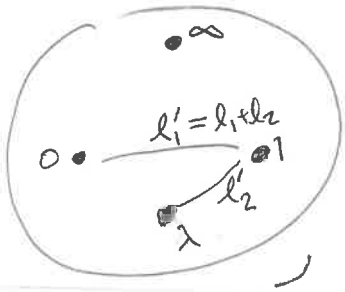


$\cong S^2$









In larger family $X = \{y^2 = (x+a)(x-b)(x-c)\}$

Pf. of Ehresmann's Thm: $f: X \to Y$: proper, submersion of smooth mfd

Ehresmann is local on target: suffices to prove for $Y = \mathbb{R}^n$

Submersions have local coordinates @ $x \in X$ given by

$$\mathbb{R}^{n+m} \xrightarrow{\text{proj}} \mathbb{R}^n .$$

Choose partition of unity for open cover of $X \Rightarrow$ lift $\partial_i \in \mathfrak{X}(\mathbb{R}^n)$

to $v_1, \dots, v_n \in \mathfrak{X}(X)$ $\left[v_i = \sum_\alpha \rho_\alpha \frac{\partial}{\partial x_i^{(\alpha)}} \right]$ s.t. $df_p(v_i) = \partial_i \ \forall p$

φ^i = flow along v_i ; $\Phi : \mathbb{R}^n \times f^{-1}(q) \to X$ by

$$\Phi(s_1, \dots, s_n, p) = \varphi^n_{s_n} \circ \cdots \circ \varphi^1_{s_1}(p)$$ is an isom.

[Uses properness to show flow is well-defined/doesn't blow up & easy to check $f(\Phi(s,p)) = s$.]

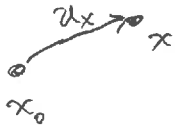
Lem: $H^c(X, \mathcal{L}) \cong M^{\pi_1(X, x_0)}$ when X is conn'd

(M = fiber of $\mathcal{L}$ @ x_0)

$$H_0(X, \mathcal{L}) \cong M_{\pi_1(X, x_0)} = M / \langle \gamma_* m - m \rangle$$

Pf: $$H_0(X, \mathcal{L}) = \left(\bigoplus_{x \in X} \mathcal{L}_x \right) \Big/ \left\langle (\alpha_* m)_{\alpha(1)} - m_{\alpha(0)} \;\middle|\; \alpha = \text{path (1-simplex)} \ \&\ m \in \mathcal{L}_{\alpha(0)} \right\rangle$$

$\phi(m_x) = u_{x,*}(m_x)$] use path to transport to x_0



$$M_{\pi_1(X, x_0)}$$

▷ ϕ is well-defined bc. $\gamma = u_x' \circ u_x^{-1}$ is loop

so $\gamma_* \circ u_{x,*}^{-1}(m_x) - u_{x,*}^{-1}(m_x) = 0$.

$\gamma_* \circ u_{x,*}^{-1}(m_x) = u_{x',*}^{-1}(m_x)$

▷ Relations exactly match after ϕ. //

Claim: $f: X \to Y$: smooth + proper $\Rightarrow$ $f_* \mathcal{L}$ is a local system if $\mathcal{L}$ is.

2 Main Ingredients:

① **Proper base change:** Suppose we have a Cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

If f is proper & $\mathcal{F}$ is constructible, then $g^* f_* \mathcal{F} \cong f'_* g^* \mathcal{F}$.

Properness used for openness of fibers & "tube lemma" [$\lim_{U \to y} \Gamma(f^{-1}U) \cong \Gamma(X_y, \mathcal{F}|_{X_y})$]

Cor:

$$\begin{array}{ccc} X_s & \to & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \xrightarrow{s} & S \end{array}$$

Then $(\mathcal{F}|_{X_s})(X_s) = f_* \mathcal{F}|_s$ for $s \in S$.

i.e. fiber of $f_* \mathcal{F}$ is $\Gamma(\mathcal{F}|_{\text{fiber}})$.

Note: proper is necessary! $f: \mathbb{A}^2 \setminus \{(0,0)\} \xrightarrow{\pi_2} \mathbb{A}^1$

$f_* \mathcal{O}_{\mathbb{A}^2 - \{0\}} \cong \mathcal{O}_{\mathbb{A}^1}[x]$ & $\Gamma(\mathcal{O}_{\mathbb{A}^1 - \{0\}}, \mathbb{A}^1 - \{0\}) = k[x, x^{-1}]$

$\Rightarrow f_* \mathcal{O}|_0 = k[x]$ ← not equal!

MORE ON BACK!

② **Ehresmann's Thm:** A proper submersion of smooth mfds is a locally trivial fibration, **ie.** $\forall y \in Y$, $\exists$ (analytic) nghd $\mathcal{U} \ni y$ s.t. $f^{-1}(\mathcal{U}) \cong \mathcal{U} \times f^{-1}(y)$.

~~By ②: $\mathcal{H}^i(X_y, \mathcal{L}|_{X_y})$ is locally trivial & ...~~

△ Can argue claim only with ② actually! ① is just needed to say $f_* \mathcal{L}$ measures $H^i(X_y, \mathcal{L}|_{X_y})$.

Prop: $f: X \to Y$: proper map & $\mathcal{L}$: local system (or constructible sheaf) on X.
Then $f_* \mathcal{L}$ is constructible on Y.

Pf: uses a powerful result on Whitney stratif^{n}'s:

★ **Thom's first isotopy lemma:** If $X = \bigsqcup_\lambda X_\lambda$ is Whitney strat^{n} s.t. $X_\lambda \to Y$ is smooth, then $f: X \to Y$ is a stratified-locally-trivial fibration. } have non sing X

★ **Existence:** Build a Whitney strat^{n} of X & Y s.t.:
- △ $f^{-1}(Y_\lambda)$ is a union of strata of X
- △ $f_\mu: X_\mu \to Y_\lambda$ is a submersion/smooth when it is defined.
- △ ($\bigsqcup_\lambda X_\lambda$ has $\mathcal{L}|_{X_\lambda}$ locally constant.)

★ Then $f_* \mathcal{L}|_{Y_\lambda}$ is a local system by isotopy lemma $\Rightarrow f_* \mathcal{L}$ is constructible //

(Co)Homology with coefficients in a local system:

for $\sigma: \Delta^i \longrightarrow X$ & $\mathcal{L}$: local system $\Longrightarrow \sigma^*\mathcal{L}$: loc. sys. on Δ^i

Note Δ^i = contractible, so $\sigma^*\mathcal{L}$ = constant shf.

$\mathcal{L}_\sigma := \sigma^*\mathcal{L}(\Delta^i)$ = fiber of $\mathcal{L}|_{\sigma(\Delta^i)}$; $\forall p \in \Delta^i$, get rest^n map

$P_p^\sigma : \mathcal{L}_\sigma \xrightarrow{\sim} \mathcal{L}_{\sigma(p)} = (\sigma^*\mathcal{L})_p$

pf that σ: $P_\tau^\sigma = (P_p^\tau)^{-1} \circ P_p^\sigma : \mathcal{L}_\sigma \longrightarrow \mathcal{L}_\tau$

$$C_i(X,\mathcal{L}) = \left\{ \sum_\sigma l_\sigma \cdot \sigma \;\middle|\; \begin{array}{l} \Sigma \text{ is finite (modify for BM)} \\ l_\sigma \in \mathcal{L}_\sigma \end{array} \right\}$$

$$\partial_i\Big(\sum_\sigma l_\sigma \sigma\Big) = \sum_\sigma \sum_{\substack{\tau: \text{face} \\ \text{of } \sigma}} \pm \underbrace{P_\tau^\sigma(l_\sigma)}_{\text{only difference from normal homology.}} \cdot \tau$$

$$H_i(X,\mathcal{L}) = \operatorname{Ker}(\partial_i) \Big/ \operatorname{Im}(\partial_{i+1})$$

Ex: $H_i(\mathbb{C}^\times, \mathcal{L}_m) = ?$ [$\mathcal{L}_m$ = local system for monodromy $z \mapsto z^m$]

e.g. $\mathbb{Z} \longrightarrow \operatorname{Aut}(A) = A^\times$, $n \longmapsto \lambda^n$

$\mathbb{C}^\times \underset{\text{homotopy}}{\sim} S^1$, so suffices to do S^1: $S^1 =$  has triangulation $T = \sigma: \Delta^1 \to S^1$

$C_1^T(\mathbb{C}^\times, \mathcal{L}_m) = C_0^T(\mathbb{C}^\times, \mathcal{L}_m) = A.$

the isomorphism matters!

$$0 \longrightarrow A \xrightarrow{\lambda - Id_A} A \longrightarrow 0$$

$$H_0(\mathbb{C}^\times, \mathcal{L}_m) = \operatorname{Ker}(\lambda_m - 1) = \begin{cases} 0 & \text{if } m \neq 1 \\ k & \text{if } m = 1 \end{cases}$$

$$H_1(\mathbb{C}^\times, \mathcal{L}_m) = k/(\lambda_m - 1) = \begin{cases} 0 & \text{if } m \neq 1 \\ k & \text{if } m = 1 \end{cases}.$$

▷ Can define similarly for IH_i

▷ Have all similar formulas as H_i: MV seqs, [Poincaré duality], if fibers are v.s.: $IH_i(X,\mathcal{L}) \cong IH^i(X,\mathcal{L})^*$

▷ $C_*(X,\mathcal{L}) = C_*(X_H) \otimes_{G/H} M$ & $C^*(X,\mathcal{L}) = \operatorname{Hom}^*_{G/H}(C_*(X_H), M)$.

$H \subset \operatorname{Ker}(\rho_x)$; M = fiber

▷ $H^0(X;\mathcal{L}) \cong M^{\pi_1(X)}$.

(19) local systems are clsd under $\oplus$, $\otimes$, $\underline{Hom}$, $(-)^\vee = \underline{Hom}(-, \underline{\Delta}_X)$

In fact, they are even better:

Prop: $Loc(X) \subset Sh(X)$ is a full abelian subcat.

Pf: $\mathcal{L} \xrightarrow{\phi} \mathcal{L}'$ a map of local systems.

$Ker(\phi)(U) := Ker(\underbrace{\mathcal{L}(U) \xrightarrow{\phi(U)} \mathcal{L}'(U)}_{\text{locally constant.}})$ ← no sheafification

$Coker(\phi)(U) := Coker[\mathcal{L}(U) \to \mathcal{L}'(U)]^{Sh}$

↖ Sheafification

Note: f_* is simple as coherent case, but f^* is not!

$f^*\mathcal{F} := f^{-1}\mathcal{F}$

$f^{-1}\mathcal{F}|_V = \underline{M}_V$ for $V = f^{-1}(U)$ s.t. $\mathcal{F}|_U = \underline{M}$

Note: NOT true for vector bdles for example:

On $\mathbb{P}^1$, have $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{pt} \to 0$.

Prop. $Loc(X)$ is stable under f^* & f_* for finite covering maps.

WARNING: Not stable under f_* for general proper maps!

(fibers can jump for example!)

Constructible Sheaves: These will capture $Rf_* \underline{k}_X$ $\forall f: X \to Y$ (proper)

Def. $\mathcal{F} \in (k_X\text{-mod})$ is constructible if $X = \coprod_\lambda X_\lambda$ is stratified so that $\mathcal{F}_{X_\lambda}$ is a local system $\forall \lambda$.

Ex: $f: \mathbb{C} \to \mathbb{C}$ by $z \mapsto z^m$

$f_* \underline{k}_{\mathbb{C}} = ?$ For D = disk not thru 0,

$$f_* \underline{k}_{\mathbb{C}}(D) = \underline{k}_{\mathbb{C}}(f^{-1}(D)) \cong k^m$$

For D = disk thru 0, $f_* k_{\mathbb{C}}(D) \cong k$.

$f_* \underline{k}_{\mathbb{C}}|_{\mathbb{C}^*} \cong \mathcal{L}_{(z \mapsto z^m \text{ on } \mathbb{C})}$ & $f_* \underline{k}_{\mathbb{C}}|_0 \cong k$



$\Rightarrow$ $k = f_* \underline{k}_{\mathbb{C}}(D) \xrightarrow{\text{res.}} f_* k_{\mathbb{C}}(D') = k^{\oplus m}$

is invts. under monodromy rep.

This example is more general than it seems!

If $f: C \to C'$ is a non-constant map btwn smooth curves, locally it is just $z \mapsto z^{e_x}$ for e_x = ramific^n index @ x.

[Ramification pts. are $x \in C'$ where f is not smooth!]

Detour: smooth maps in algebraic geometry.

"Smoothness of a map $f: X \to Y$" $\iff$ "submersion of (pseudo)manifolds"

Def: $\operatorname{Spec}(A[x_1,\dots,x_n]/(g_1,\dots,g_r)) \to \operatorname{Spec}(A)$ is smooth if $\left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{1\le i\le r \\ 1\le j\le n}}$ is of rank r $\forall x$.

i.e. $f: X \to Y$ is smooth iff it induces surjections on tangent spaces.

non-ex: $\pi: \operatorname{Spec} k[u,v] \to \operatorname{Spec} k[x,y]$ by $x \mapsto u$, $y \mapsto uv$.

$k[u,v] \cong k[x,y][v]/(xv-y)$ [x] vanishes @ $x=0$.

Thm: $\varphi: X \to Y$ is smooth iff φ is flat, locally of finite presentation and its fibers are smooth.

E.g. • $\varphi: \mathbb{C} \to \mathbb{C}$ by $z \mapsto z^m$

Fiber @ 0 is $\mathbb{C}[x]/(x^m)$ ← non-reduced, so not smooth!

• Elliptic curve ex. is smooth on $\mathbb{A}^1 - \{0,1\}$ but not if we extend to 0 or 1