(13) # Poincaré Duality for IH:

Recall: $\overline{t}(i) = \begin{cases} 0, & i \le 2 \\ i-2, & i > 2 \end{cases}$ is the maximal/top perversity and

$\underline{0}(i) = 0 \ \forall i$ is the zero perversity.

▷ For p: perversity fcn, define $q = \overline{t} - p$ to be the "complementary perversity".

Thm: X: projective complex variety; p & q = complementary. Then there is a non-degenerate bilinear pairing

$$IH^p_i(X;\mathbb{Q}) \times IH^q_{n-i}(X,\mathbb{Q}) \xrightarrow{\cap} \mathbb{Q}.$$

Notes: ▷ similar versions when X is not projective adding suitable "cpt. supports" conditions.

▷ when X is smooth this is constructed by taking $\sigma_i : \Delta^i \to X$ & $\sigma_{n-i} : \Delta^{n-i} \to X$, nudging them until they are transverse, & sending to $\#(\sigma_i \cap \sigma_{n-i})$. Proof of non-degeneracy comes from local computations + M.V. sequence

Idea for proof is similar: take chains s.t. $|\sigma_i| \cap |\sigma_{n-i}| \subseteq X^{sm}$ & is finitely many points. Using M.V., reduces to showing the corresponding statement for cones $\text{cone}(L)$.

Recall: for $\dim(L) = m$,

$$IH^p_i(\text{cone}\,L, \mathbb{Z}) = \begin{cases} IH^p_i(L;\mathbb{Z}), & i < m - p_{m+1} \\ 0, & \text{else} \end{cases}$$

$$I^{BM}H^p_i(\text{cone}\,L, \mathbb{Z}) = \begin{cases} 0, & i \le m - p_{m+1} \\ IH^p_{i-1}(L;\mathbb{Z}), & \text{else} \end{cases}$$

(copy before class)

If L satisfies P.D. (induction on dim), then $\forall i < m - p_{m+1}$

$$IH^p_i(\text{cone}\,L, \mathbb{Q}) \cong IH^p_i(L, \mathbb{Q}) \xrightarrow{\cong} IH^q_{n-i}(L, \mathbb{Q})^* \cong I^{BM}H^q_{m+1-i}(\text{cone}\,L, \mathbb{Q})^*$$

& if $i \ge m - p_{m+1}$, since $m - 1 = p_{m+1} - q_{m+1}$,

$$IH^p_i(\text{cone}\,L, \mathbb{Q}) = 0 = I^{BM}H^q_{m+1-i}(\text{cone}\,L, \mathbb{Q})$$

Later, we'll return & reprove this with sheaves / Verdier duality.

(13.5) Rmk: Here, the complentary perversities appear in local calculations (for cones). Another way to see complementary perversity fcns must appear is through the heuristics of intersection #'s:

an allowable i-chain σ_i & $n-i$ chain σ_{n-i} must not meet on X_{n-k}. Their expected intersection dim. is

$$(i-k+p_k)+(n-i-k+q_k)-(n-k)=p_k+q_k-k.$$

To make this <0, need $p_k+q_k \leq k-1$

For homologies to also miss each other: $p_k+q_k \leq k-2$

Making this equal allows "as many cycles in dual as possible" $\longrightarrow$ makes pairing non-degenerate.

(14) Note there a 2 "middle" perversities:

$\underline{m} = [0,0,0,1,1,2,2,3,3,\dots]$ "upper middle"

$\underline{n} = [0,0,0,0,1,1,2,2,3,3,\dots]$ "lower middle"

$m + n = \underline{t}$ are complementary.

For general pseudomanifolds (locally conical mfds), $IH^{\underline{m}} \neq IH^{\underline{n}}$

e.g. real algebraic varieties don't necessarily have $IH^{\underline{m}} = IH$

Prop. If all strata in a Whitney stratifn of X are even dimensional, then $IH^{\underline{m}}_*(X) = IH^{\underline{n}}_*(X)$.

ex: X: complex algebraic variety.

Pf: Since strata are even-dim'l, they only see $p(2i)$ for $0 \le i \le \dim_{\mathbb{C}}(X) = \frac{1}{2}\dim_{\mathbb{R}}(X)$. //

In this case, $IH^{\underline{m}}_*(X) = IH_*(X)$ is self-dual under Poincaré Duality.

e.g. If X: cx alg. variety has smooth normalization, then $IH^{\underline{p}}_*(X) = IH_*(X)$ $\forall$ perversity fcns p & so $IH^{\underline{p}}_*(X)$ satisfy Poincaré Duality.

Introduces new invariants associated to X

e.g. If X has even complex dimension $2n$, get a non-deg. pairing $IH^{\underline{m}}_{2n}(X,\mathbb{Q}) \otimes IH^{\underline{m}}_{2n}(X,\mathbb{Q}) \longrightarrow \mathbb{Q}$.

$\sigma(X) :=$ #(positive eigvals) $-$ #(negative eigvals)

"signature of X" $\longrightarrow$ topological (bordism) invariant of X.