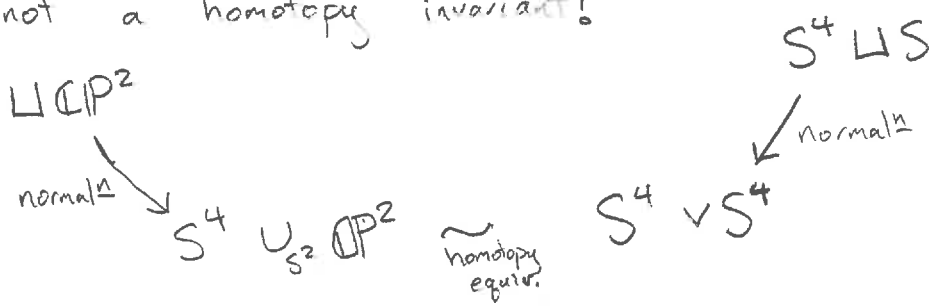
Rmk: IH is not a homotopy invariant!

e.g. $S^4 \sqcup \mathbb{CP}^2$ and $S^4 \sqcup S^4$



$$\Rightarrow IH_i^{\bar{p}}(S^4 \vee S^4, \mathbb{Z}) \cong H_i(S^4, \mathbb{Z}) \oplus H_i(S^4, \mathbb{Z}) \quad \text{different!}$$
$$IH_i^{\bar{p}}(S^4 \cup_{S^2} \mathbb{CP}^2, \mathbb{Z}) \cong H_i(S^4, \mathbb{Z}) \oplus H_i(\mathbb{CP}^2, \mathbb{Z})$$

Note: In both formulas above, answer is independent of $\bar{p}$! This reflects smoothness of normalization, as we prove below:

Thm. X: oriented, normal, cpt n-dim'l pseudomfd / X: projective, normal (real dim n) complex **variety**.

Then: $IH_i^{\bar{t}}(X, \mathbb{Z}) \cong H_i(X, \mathbb{Z})$
$IH_i^{\bar{0}}(X, \mathbb{Z}) \cong H^{n-i}(X, \mathbb{Z})$

[$\bar{0}$ = 0-perversity fxn; $\bar{t}$ = top perversity fxn, $\bar{t}(i) = i-2$, $i \geq 2$. — Other perversities "interpolate" between H_i & H^{n-i}.]

Pf Ideas: **LemA:** $IH_0^{\bar{p}}(Y) = H_0(Y^{sm})$

[Compute w/ chains: $\dim(|\xi| \cap Y_{n-2}) \leq -2$ for $\xi \in IC_0^{\bar{p}}$ & $\dim(|\xi| \cap Y_{n-2}) \leq -1$ for $\xi \in IC_1^{\bar{p}}$

$\Rightarrow IH_0^{\bar{p}}$ same as computing H_0 for $Y^{sm} = Y - Y_{n-2}$.]

MOVE TO LATER

LemB L: compact of dim. $k-1$, with Whitney strat^n $L = L_{k-1} > L_{k-2} > \cdots > L_0$

$$IH_i^{\bar{p}}(\text{cone}(L)) = \begin{cases} IH_i^{\bar{p}}(L), & i < k-1-p_k \\ 0, & i \geq k-1-p_k \end{cases}$$

Pf: $\text{cone}(L)$ has stratif^n $\text{cone}(L)_i = \begin{cases} \text{cone}(L_{i-1}), & i \geq 1 \\ \{v\}, & i = 0 \end{cases}$ (cone point)

$\xi \in C_i(\text{cone}L)$ is allowable at v iff:

$0 \leq i - k + p_k \iff i \geq k - p_k$

If $i \leq k - p_k - 1$, $\xi \in IC_i^{\bar{p}}(\text{cone}L)$ can't touch v &

$$IC_i^{\bar{p}}(\text{cone}(L)) \cong IC_i^{\bar{p}}(L \times (0,1)) \cong IC_i^{\bar{p}}(L) \Rightarrow \text{for } i < k-1-p_k,\ IH_i^{\bar{p}}(\text{cone}L) = IH_i^{\bar{p}}(L)$$

If $i \geq k - p_k - 1$, form $c\xi = \mathrm{cone}(\xi)$ for $\xi \in IC^p_i(\mathrm{cone}\, L)$

By hypothesis, $\dim(|c\xi| \cap \{v\}) = 0 \leq (i+1) - k + p_k$

& $\dim(|c\xi| \cap \mathrm{cone}(L_{k-1-j})) \leq \dim(|\xi| \cap L_{k-1-j}) + 1$

$\leq i - j + p_j + 1$

So $c\xi \in IC^p_{i+1}(\mathrm{cone}\, L)$.

Easy to check: $\partial(c\xi) = \xi \Rightarrow IH^p_i(X) = 0$ for $i \geq k - p_k - 1$ //

Lem: $IH^p_*(\mathbb{R}^m \times Y) \cong IH^p_*(Y)$

[Pass to normalizations & split as product. Use $H_*(\mathbb{R}) \cong \mathbb{R}$ in deg 0

Mayer-Vietoris for IH: For $U, V \subseteq X$ open, we have a s.e.s of complexes:

$$0 \to IC^p_*(U \cap V) \xrightarrow{\xi \mapsto (\xi, \xi)} IC^p_*(U) \oplus IC^p_*(V) \xrightarrow{(\alpha,\beta) \mapsto \alpha - \beta} IC^p_*(U \cup V) \to 0$$

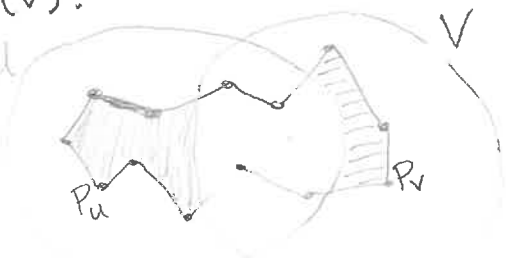
[so we have a les.

$$\cdots \to IH^p_i(U \cap V) \to IH^p_i(U) \oplus IH^p_i(V) \to IH^p_i(U \cup V) \xrightarrow{\partial} IH^p_{i-1}(U \cap V) \cdots$$

Pf: Most is formal from usual MV sequence.

Hardest Part: If $\xi \in IC^p_i(U \cup V)$, then $\xi = \xi_U + \xi_V$ for $\xi_U \in IC^p_i(U)$ & $\xi_V \in IC^p_i(V)$.

Pf Sketch:



Shrink simplices enough s.t. no simplex intersects both P_U & P_V. AND $|\xi| \cap X_i$ are sub-complexes

Pick $c \in (0,1)$ avoiding vertices, & ξ_U = part of ξ w/ $f < c$ & ξ_V = part of ξ w/ $f > c$.

Compatibility is clear for $|\xi_U|$ & $|\xi_V|$.

$|\partial \xi_U| \subseteq |\partial \xi| \cup f^{-1}(c)$

define $P = |\xi|$

$f: P \to \mathbb{R}$ cts. s.t.

$f(v) = \begin{cases} 0 & \text{if } v \in \{\text{clsd simplices intersecting } P_U\} \\ 1 & \text{if } v \in \{\text{clsd simplices intersecting } P_V\} \\ \frac{1}{2} & \text{else} \end{cases}$

for vertices v & extending affinely to P.

"Thin Wall Lemma": For every subpolyhedra $Y \subseteq P$, $\dim(f^{-1}(c) \cap Y) \leq \dim(Y) - 1$ //

(12) # Pf of $IH^{\bar{p}}$ & $IH^{\bar{t}}$ for normal X:

Claim: If F & G : $\{\text{open sets in } X\} \longrightarrow \{\text{graded abelian gps}\}$ both satisfy:

① $F(U) \longrightarrow F(V)$ for $U \hookrightarrow V$

② F & G satisfy M.V. exact sequences

③ F & G commute w/ direct limits over increasing unions.

Then if $\phi : F \longrightarrow G$ is a natural transformation (which is an isomorphism when $V \cong \mathbb{R}^n \subset X^{sm}$ or $V \cong \mathbb{R}^{n-k} \times \text{cone}(L)$, then ϕ_X is an isomorphism.

Pf: Use MV sequences: • if ϕ is an isom. on U, V, $U \cap V$, then $\phi_{U \cup V}$ is isomorphism.

• Induct on Whitney stratification, covering X_{n-i} at each step.

Thm Part 1: $IH_i^{\bar{t}}(X, \mathbb{Z}) \cong H_i(X, \mathbb{Z})$

Pf: need to check for cone(L):

i	$IH_i^{\bar{t}}(\text{cone}\,L)$	$H_i(\text{cone}\,L)$
≥ 1	0	0 [cone L is contractible]
0	$IH_0^{\bar{t}}(L) = H_0(L^{sm})$	$\mathbb{Z}$

$\dim(|\xi| \cap X^{sing}) \leq i-2$ $\forall$ coh degs. k. $\Rightarrow IC_*^{\bar{t}} \leq C_*^{PL}$ $\Rightarrow IH_*^{\bar{t}} \longrightarrow H_*$

$k-1-(k-2) = 1$, so follows from Lems B, then A

Agree when L^{reg} is connected $\Longleftarrow$ top. normality. //

Thm Part 2: $IH_i^{\bar{p}}(X, \mathbb{Z}) \cong H^{n-i}(X; \mathbb{Z})$

map comes from $\sigma \longmapsto D(\sigma)$ = dual block of barycentric subdivision.

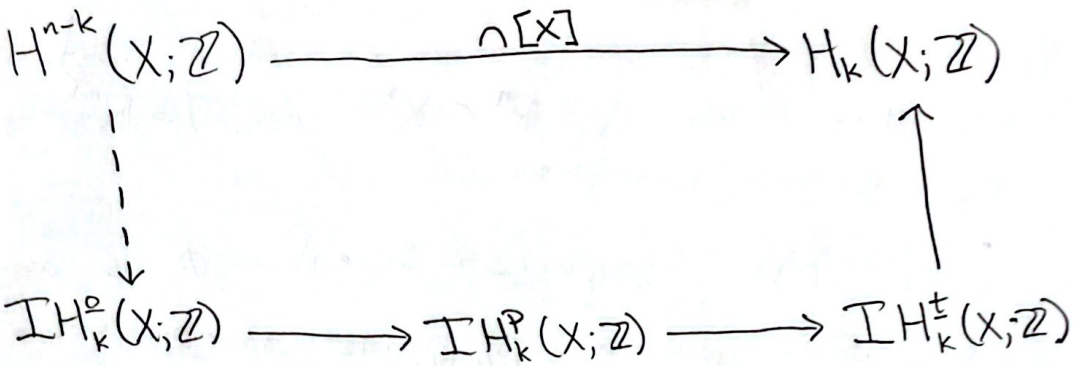
↑ This map requires starting with a triangulation! Better: use the cap product. See Buck.

$V = \mathbb{R}^{n-k} \times \text{cone}(L)$:

i	$IH_i^{\bar{p}}(V)$		$H_c^{n-i}(V)$
$\leq k-2$	$IH_i^{\bar{p}}(L)$	← Induction →	$H^{k-1-i}(L)$
$k-1$	0	← when L is connected, i.e. normality →	$\tilde{H}^0(L)$
$\geq k$	0		0

$k-1-0 = k-1$, so Lem B computes //

Prop. [Cap product factors through IH]: $\forall$ perversity fcn $\bar{p}$, there is a factorization



- Can be proved using a triangulation of X + explicit formula for cap product.
- We will defer proof as it follows more cleanly from sheaf-theoretic constructions.

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(12.5) Borel-Moore Homology & $I^{BM}H$:

(-Introduce/motivate BM homology
-MV for BM
-Compute for cones.

X: smooth, but non-compact, oriented mfds.

→ duality & many applications of cohomology require compact supports

eg. $H^i_c(X,\mathbb{Z}) \cong H_{n-i}(X,\mathbb{Z})$

$H^i(X,\mathbb{Z}) \cong H^{BM}_{n-i}(X,\mathbb{Z})$

"Borel-Moore" → generalizes well to IH

Borel-Moore Homology: Replaces singular chains $\sum_{i=1}^{k} a_k \cdot \sigma_k$ with infinite sums $\sum a_\sigma \cdot \sigma$ s.t. $\forall$ compact $K \subseteq X$, only finitely many σ intersect K.

Ex: for $\mathbb{R}$, we have $\sum_{i\in\mathbb{Z}} [i,i+1] \in C_1((|\mathbb{R}|))$

Can use a triangulation: $\mathbb{R} = \bigcup_{i\in\mathbb{Z}} [i,i+1]$.

$$0 \longrightarrow \prod_{n\in\mathbb{Z}} \mathbb{Z} \longrightarrow \prod_{n\in\mathbb{Z}} \mathbb{Z} \longrightarrow 0$$

$$(a_n) \longmapsto (a_{n+1}-a_n)$$

$H_1 = \mathbb{Z}\cdot(1)_{n\in\mathbb{Z}}$ $\quad H_0 = 0$

Fact: If $\overline{X}$ is cpt'n of X & $D = \overline{X} - X$,

$H^i_c(X,\mathbb{Z}) \cong H^i(\overline{X}, D; \mathbb{Z})$

If M is compact, it is clear (from chains) that $H_i = H^{BM}_i$.

Can do the same modification to get $I^{BM}H^p_i$. All results we discussed for IH hold for $I^{BM}H$ (compute by normalizations, MV sequences, etc.).

Goal: $I^{BM}H^p_i(\underbrace{cL}_{\text{cone}(L)}) = ?$ for $\dim(L) = k-1$; L: compact (w/o bdry) pseudomfd.

SubLem: $I^{BM}H^p_i(X;\mathbb{Z}) \cong I^{BM}H^p_{i+1}(X\times\mathbb{R};\mathbb{Z})$

Pf: Still splits as a product, but $I^{BM}H^p_i(\mathbb{R},\mathbb{Z}) = \begin{cases} 0 & , i=0 \\ \mathbb{Z} & , i=1 \end{cases}$

Prop $I^{BM}H^p_i(cL,\mathbb{Z}) = \begin{cases} IH^p_{i-1}(L,\mathbb{Z}) & , i > k-1-p_k \\ 0 & , i \le k-1-p_k \end{cases}$ ↩ inequalities reversed from standard case!

Pf: Same condition at cone point: $\dim(|\xi| \cap \{0\}) \le i-k+p_k$

so no intersection when $i \le k-1-p_k$ [when $i-k+p_k \le -1$]

For $\xi \in IC^p_i(L\times(0,\infty))$ (missing 0) form $R = \sum_{m\ge 0} [m,m+1]$ on $[0,\infty)$ & $\xi\times R$.

& consider $\eta = F_*(\xi\times R)$: composing $\xi\times R$ with $F: L\times(0,\infty)\times[0,\infty) \xrightarrow{\pm} L\times(0,\infty)$

$\partial(\xi\times R) = \partial\xi \times R + (-1)^i \xi \times \partial R \overset{\partial\xi=0}{=} (-1)^{i+1} \xi \times \{0\} \Rightarrow \partial\eta = \xi$

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So this "coning off to infinity" procedure shows

$$I^{BM}H_i^P(cL) = 0 \quad \text{for} \quad i \leq k-1-p_k.$$

For higher degrees $\xi \in IC_i^P(L) \Rightarrow c_\infty(\xi) = \overline{\xi \times \mathbb{R}} \approx \xi \times [0,\infty)$

$$c_\infty(\xi) \in I^{BM}C_{i+1}^P(L)$$

$$c_\infty : IC_*^P(L) \hookrightarrow IC_{*+1}^P(cL)$$

for $*$ large enough to go through cone pt.

$$\partial(c_\infty \sigma) = -c_\infty(\partial\sigma)$$

//