⑥ **Cor.** X = pinched torus $= \{y^2 = x^3 - x^2\} \subseteq \mathbb{P}^2$ has

$$IH_i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & i = 1 \end{cases}$$

Pf: $H_1(X - x, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_1(X, \mathbb{Z})$

($X - x \simeq S^1$)

$1 \longmapsto 0$

bc gen. is a bdry in X



//

~~Will need a perversity function at all~~ ← Will return to this later

---

- Computing with chains is hard; often computed via normalizations.

Normalization: "a type of partial compactification"

Def. A domain A is *normal* if it is integrally closed in its field of fractions.

i.e. $F = \mathrm{Frac}(A) \supseteq A$. A is normal if for every $a_1, \dots, a_m \in A$ & every $x \in F$ s.t. $x^m + a_1 x^{m-1} + \cdots + a_m = 0$ we have $x \in A$.

Def. The normalization of A is

$$\{x \in F \mid x \text{ satisfies } x^m + a_1 x^{m-1} + \cdots + a_m = 0 \text{ for some } m \text{ \& } a_i \in A\}.$$

⑦ Ex: $\{x^2 = y^3\} \subseteq \mathbb{C}^2$ (has singularity at (0,0))

$\Updownarrow$

$A = \mathbb{C}[x,y]/(x^2-y^3) \cong \mathbb{C}[t^3, t^2] \subseteq \mathbb{C}[t]$

$\mathrm{Frac}(A) \cong \mathbb{C}(t)$

t satisfies $t^3 - (x) = 0$ & $\overline{A} = \mathbb{C}[t]$

$A \hookrightarrow \overline{A} \iff \{x^2 = y^3\} \leftarrow \mathbb{A}^1$

(not equiv. to resolutions of singularities)

Normality is related to resolutions of singularities

Prop: If A is normal (noetherian), then A is "regular in codim 1"

ie. for every height 1 prime $\wp \subseteq A$, $A_\wp$ is regular.

(Geometrically, the singular locus of A is codim ≥ 2.)

Pf: Since $A_\wp$ is normal, it is sufficient to show:

Claim: If A is a normal, noetherian local ring of dimension 1, then its maximal ideal $m \subset A$ is principal.

Pf of Claim: $x \in m$ nonzero $\Rightarrow m = \sqrt{(x)} \Rightarrow m^n \subset (x)$ some n

Choose n smallest s.t. $m^n \subseteq (x)$, so $\exists y \in m^{n-1}$ s.t. $y \notin (x)$

$\Rightarrow t = \frac{y}{x} \in \mathrm{Frac}(A) \setminus A$ & $t \cdot m \subset A$ so $tm = A$ or $\subset m$.

If $tm \subset m \Rightarrow tm$ is f.g. A-module & is $A[t]$-module

$\Rightarrow t$ is integral /A $\Rightarrow$ contradiction.

So $t \cdot m = A$ & $\exists n \in m$ s.t. $tn = 1 \Rightarrow m = A \cdot t^{-1} = A \cdot n$

is principal //

So the normalization is a "partial resolution".

What does normality look like?

Def. X is topologically normal if $\forall x \in X$, $\exists$ open nghd $U_x \ni x$ s.t. $U_x - X_{n-2} = U_x - \mathrm{Sing}(X)$ is connected.

Ex:  is not topologically normal, but  is.

⑧ **Prop.** X normal $\Longrightarrow$ X topologically normal.

Pf: We use a key property of normal varieties:

Zariski's Main Thm: If $f: Z \to X$ is a proper, birat'l map; X is normal & Z is irreducible, then ① $f^{-1}(x)$ is connected $\forall x \in X$, ② $f_* \mathcal{O}_Z = \mathcal{O}_X$.

Hironaka's Thm: Any quasiprojective variety X admits a (proper) resolution of singularities $f: Z \to X$.

We argue by passing to Z: $x \in U$ (analytic) nghd of x. Then put $E = \pi^{-1}(U) \cap \pi^{-1}(\mathrm{Sing}(X))$. E is clsd in Z & Z is smooth, so $\pi^{-1}(U) \setminus E$ is still connected if $\pi^{-1}(U)$ is connected (in analytic top.) (bc $\pi^{-1}(U) = V_1 \sqcup V_2 \Rightarrow \pi(V_1) \cap \pi(V_2) = \emptyset$ (because f is a resolution of singularities) $\Rightarrow U = \pi(V_1) \sqcup \pi(V_2)$) //

$\| 2$

$U \setminus \pi(U)$

Thm (Goresky–MacPherson) $\pi: \tilde{X} \to X$: the normalization of X. Then $\pi_*: IH^{\bar{p}}_i(\tilde{X}; \mathbb{Z}) \to IH^{\bar{p}}_i(X; \mathbb{Z})$ is an isom.

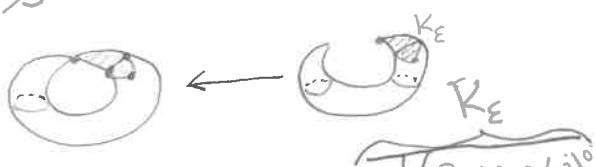
Pf: π induces a map on complexes $\pi_*: IC^{\bar{p}}_i(\tilde{X}, \mathbb{Z}) \to IC^{\bar{p}}_i(X;$ which takes a chain $\sigma: \Delta^i \to \tilde{X}$ to $\pi \circ \sigma$.

If $X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$ is strat of X, we can take preims. to get $\tilde{X}_n \supseteq \tilde{X}_{n-1} \supseteq \cdots \supseteq \tilde{X}_0$, which suffices to compute $IC^{\bar{p}}_i(\tilde{X};$ (Exercise: $\tilde{X}_n \supseteq \cdots \supseteq \tilde{X}_0$ is a Whitney stratification) ← See Back

We claim any chain $\sigma: \Delta^i \to X$ lifts uniquely to $\tilde{X}$

Since $\sigma(\text{Interior of } \Delta^i) \subseteq X^{\text{smooth}}$ & π is isom. over X^{smo} there is unique:

$$\begin{array}{ccc} (\Delta^i)^\circ & \xrightarrow{\tilde{\sigma}^\circ} & \tilde{X} \\ \cap & & \downarrow \\ \Delta^i & \xrightarrow{\sigma} & X \end{array}$$



For $a \in \Delta^i$, $B_\varepsilon(a) \cap (\Delta^i)^\circ$ is conn'd, so $\pi^{-1}(B_\varepsilon(a) \cap (\Delta^i)^\circ)$ is connected. $K_\varepsilon \subseteq \pi^{-1}(\sigma(\Delta^i))$: cpt (bc π is proper) $\subseteq \pi^{-1}$ (for any pair)

$\bigcap_{\varepsilon > 0} K_\varepsilon$ is non-$\emptyset$ & conn'd (bc they are nested $\neq \emptyset$ compact sets) ⇝ so & intersection is a pt. & can lift uniquely &

↑ not true for arbitrary resolutions of singularities

8.5

Note on functoriality of IH:

Prop. Suppose $f: Y \longrightarrow X$ is a finite map. Then there are Whitney stratifications

$$Y = Y_n > Y_{n-1} > \cdots > Y_0 \quad \& \quad X = X_n > X_{n-1} > \cdots > X_0$$

s.t. $Y_i = f^{-1}(X_i)$.

Pf: Build stratifications together.

At each stage of induction, remove points, their preimages, & their images whenever they fail the 2 Whitney conditions. As this is a closed condition, we can then continue the induction downwards on dimension. //

This has applications beyond normal varieties!

e.g. $q: Y \longrightarrow Y/G$ for a finite group G acting on a normal variety Y.

Exercise: Show $IH^p_*(Y/G, \mathbb{Q}) = IH^p_*(Y, \mathbb{Q})^G$.

(9) **Ex:** Back to Whitney umbrella:

$X = \mathrm{Spec}(A) = \{x^2 = y^2 z\} \subseteq \mathbb{C}^3$

$$A = \mathbb{C}[x,y,z]/(x^2 - y^2 z) \hookrightarrow \mathbb{C}[u,v]$$

$x \mapsto uv$

$y \mapsto u$

$z \mapsto v^2$

$A \cong \mathbb{C}[u, uv, v^2]$ ← (Can get v from $\lambda^2 - (v^2)$)

$\overline{A} = \mathbb{C}[u,v]$

$IH(X;\mathbb{Z}) = IH(\mathbb{C}^3;\mathbb{Z})$

For projective varieties, can glue normalizations for affine charts.

e.g. $X = \{x^2 w = y^2 z\} \subseteq \mathbb{P}^3$

On $w \neq 0$: $\nu_w : \mathbb{A}^2_{u,v} \to X$, $(u,v) \mapsto [uv : u : v^2 : 1]$

On $z \neq 0$: $\nu_z : \mathbb{A}^2_{a,b} \to X$, $(a,b) \mapsto [a : ab : 1 : b^2]$

glued where $v \neq 0$ / $b \neq 0$ with

$$\left[\frac{u}{v} : \frac{u}{v^2} : 1 : \frac{1}{v^2}\right] = [a : ab : 1 : b^2]$$

$a = \frac{u}{v}$, $b = \frac{1}{v}$

$\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1))$ $\mathbb{P}^1$ gluing

+ $w = z = 0$ compactifies $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1))$

$\Rightarrow \overline{X} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ "Hirzebruch surface"

$$IH_i(X,\mathbb{Q}) = H_i(\overline{X},\mathbb{Q}) \overset{\text{projective bundle formula}}{\underset{\text{[See back]}}{\cong}} H_i(\mathbb{P}^1,\mathbb{Q}) \oplus H_{i-2}(\mathbb{P}^1,\mathbb{Q})$$

$$= \begin{cases} \mathbb{Q}, & i=0 \\ 0, & i=1 \\ \mathbb{Q}\oplus\mathbb{Q}, & i=2 \\ 0, & i=3 \\ \mathbb{Q}, & i=4 \end{cases}$$

Projective Bundle formula: $\mathcal{E} \to X$: vector bundle.

$$H^*(\mathbb{P}(\mathcal{E}); \mathbb{Z}) \cong \frac{H^*(X;\mathbb{Z})[\xi]}{\left(\xi^{r+1} + c_1(\mathcal{E})\xi^r + c_2(\mathcal{E})\xi^{r-1} + \cdots + c_{r+1}(\mathcal{E})\right)}$$

As $H^*(X;\mathbb{Z})$-algebras

Cor: $$H^k(\mathbb{P}(\mathcal{E}); \mathbb{Z}) \cong \bigoplus_{i=0}^{r} H^{k-2i}(X;\mathbb{Z}) \quad \forall k.$$