LECTURE 2:

Intersection Homology: X: singular $\mathbb{C}$-variety.

0: Sings. are interesting. e.g. $x^n = y^m$

I: Whitney Stratifications

$X = \coprod_\lambda X_\lambda$ stratification (ie. X_λ connd, locally clsd, smooth, $\overline{X_\lambda} = \bigcup_\mu X_\mu$)

We ask X_λ is "equisingular", i.e. $\forall x \in X_\lambda$, $\exists$ nghd $U \ni x$ s.t. $(X_\lambda \cap U) \times V \cong U$ (id on $X_\lambda \cap U$) ("local normal triviality")

Ex: Whitney umbrella $\{x^2 = zy^2\} \subseteq \mathbb{C}^3$.

$\mathbb{C}^2 \to$  $(u,v) \mapsto (uv, v, u^2)$

$X_{reg} \cup X_{sing}$ not good enough!

$X_{reg} \cup (X_{sing} \setminus 0) \cup \{0\}$

$y^2 + x^3 - t^2 x^2 = 0$: 

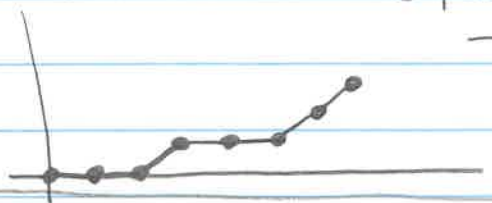
Whitney Condition: $X_\mu \subset \overline{X_\lambda} \Rightarrow \forall a_i \in X_\lambda$, $b_i \in X_\mu$ converging to $c \in X_\mu \to \lim_{i\to\infty}$ (secants $a_i - b_i$) $\in \lim_{i\to\infty} T_{a_i} X$

Intersection Homology: to recover P.D. / intersection thy, need to restrict what cycles we deal with.

Def: A perversity is a fxn $\bar{p} : [0, \dim X] \cap \mathbb{Z} \to \mathbb{N}$ (real dimension) s.t. $p_\alpha := \bar{p}(\alpha)$ satisfies $\begin{cases} p_0 = p_1 = p_2 = 0 \\ p_\alpha \le p_{\alpha+1} \le p_\alpha + 1, \ \alpha \ge 2 \end{cases}$

p. 63

e.g. 

 } slopes 0 or 1 everywhere.

e.g. upper middle perversity: $(0,0,0,1,1,2,2,\dots,\frac{n}{2}-1)$ ★ DEFAULT

lower middle: $(0,0,0,0,1,1,2,2,\dots,\frac{n}{2}-1,\frac{n}{2}-1)$

$X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$ $(\dim X_i \le i)$

An i-chain $\xi \in C_i(X)$ is $\bar{p}$-allowed if $\dim(|\xi| \cap X_{n-\alpha}) \le i - \alpha + p_\alpha$

Note: $\xi \pitchfork X_{n-\alpha} \iff \dim(|\xi| \cap X_{n-\alpha}) = i - \alpha$

$$IC_i^{\bar{p}}(X) = \left\{ \xi \in C_i(X) \;\middle|\; \begin{array}{l} \dim(|\xi| \cap X_{n-\alpha}) \le i - \alpha + p_\alpha \\ \dim(|\partial\xi| \cap X_{n-\alpha}) \le i - 1 - \alpha + p_\alpha \end{array} \ \forall \alpha \ge 2 \right\}$$

$$IH_i^{\bar{p}}(X; \mathbb{Z}) = H_i\left(IC_\bullet^{\bar{p}}(X)\right)$$

⑤ • Ex: $X = \mathbb{P}^1 \cup \mathbb{P}^1 \subseteq \mathbb{P}^2$, $X_2 = \mathbb{A}^1 \cup \mathbb{A}^1 \supseteq X_1 = X_0 = \{pt\}$

$p_0 = p_1 = p_2 = 0$ is only option for perversity fxn.

$$IC_i(X) = \left\{ \xi \in C_i(X) \;\middle|\; \begin{array}{l} \dim(|\xi| \cap X_0) \leq i-2 \\ \dim(|\partial\xi| \cap X_0) \leq i-3 \end{array} \right\}$$

Chains can only touch singularity at $i=2$ (impossible due to this)

Chain boundaries can never touch the singularity

$$\Rightarrow IC_\bullet(X) \simeq C_\bullet(\mathbb{P}^1 \sqcup \mathbb{P}^1)$$

$$\& \quad IH_i(X) = H_i(\mathbb{P}^1 \sqcup \mathbb{P}^1) = \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0, 2 \\ 0 & i = 1. \end{cases}$$

More generally:

• Ex: X: $2m$-dimensional with an isolated singular point x

$$X = X_{2m} \supset X_0 = \{x\}.$$

Let's use the middle perversity

Prop: $$IH_i(X, \mathbb{Z}) = \begin{cases} H_i(X, \mathbb{Z}), & i > m \\ \mathrm{Im}(H_m(X-x, \mathbb{Z}) \to H_m(X, \mathbb{Z})), & i = m \\ H_i(X-x, \mathbb{Z}), & i < m \end{cases}$$

Pf: $\xi \in IC_i^{\underline{m}}(X)$ iff ① $\dim(|\xi| \cap X_0) \leq i - 2m - \underline{m}(2m) = i-m-1$
② $\dim(|\partial\xi| \cap X_0) \leq i-m-2$

If $i \leq m$, can't intersect $X_0 = \{x\} \Rightarrow IC_i^{\underline{m}}(X) = C_i(X-x)$

If $i \geq m+2$, no condition $\Rightarrow IC_i^{\underline{m}}(X) = C_i(X)$

If $i = m+1$, ξ can go through x, but not $\partial\xi$

$$IC_\bullet(X): \cdots \to C_{m+3}(X) \to C_{m+2}(X) \to IC_{m+1}(X) \to C_m(X-x) \to C_{m-1}(X-x) \to \cdots$$

Result is clear for $i \neq m, m+1$.

$$\mathrm{Ker}\left[IC_{m+1}(X) \to C_m(X-x) \right] = \mathrm{Ker}\left[C_{m+1}(X) \to C_m(X) \right]$$

($IC_{m+1}(X) \cap C_{m+1}(X)$; since if $\partial\xi = 0$, $x \in |\partial\xi|$)

$\Rightarrow$ result holds at $m+1$

$$\partial : IC_{m+1}(X) \to \partial C_{m+1}(X) \cap \underbrace{IC_m(X)}_{C_m(X-x)}$$

$\Rightarrow IH_n$ is image of $H_n(X-x)$ in $H_n(X)$ //

⑥ **Cor.** X = pinched torus $= \{y^2 = x^3 - x^2\} \subseteq \mathbb{P}^2$ has

$$IH_i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & i = 1 \end{cases}$$

Pf: $H_1(X - x, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_1(X, \mathbb{Z})$

($X - x \simeq S^1$)

$1 \longmapsto 0$

bc gen. is a bdry in X



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~~Will need a perversity function at all~~ ← Will return to this later

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- Computing with chains is hard; often computed via normalizations.

Normalization: "a type of partial compactification"

Def. A domain A is *normal* if it is integrally closed in its field of fractions.

i.e. $F = \mathrm{Frac}(A) \supseteq A$. A is normal if for every $a_1, \dots, a_m \in A$ & every $x \in F$ s.t. $x^m + a_1 x^{m-1} + \cdots + a_m = 0$ we have $x \in A$.

Def. The normalization of A is

$$\{x \in F \mid x \text{ satisfies } x^m + a_1 x^{m-1} + \cdots + a_m = 0 \text{ for some } m \text{ \& } a_i \in A\}.$$