(3.5)

Recap of some algebraic topology:

Chains: Δ^n = n-simplex

$C_n(X)$ = free abelian group generated by maps $\Delta^n \longrightarrow X$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \cdot (\sigma \circ d_i)$$ for $d_i : \Delta^{n-1} \hookrightarrow \Delta^n$ the i^{th} face.

$$H_i(X) = H_i(\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots)$$

Rmk: Can replace $C_n(X)$ with set of (possibly infinite) sums $\sum c_i \sigma_i$ s.t. every cpt $K \subset X$ intersects only finitely many σ_i. In this case, the complex computes Borel-Moore homology, a sort of "homology w/cpt supports".

e.g. $\sigma_i : \Delta^1 \longrightarrow \mathbb{R}$ by the interval $[i, i+1]$

$$\sum_{i \in \mathbb{Z}} \sigma_i \in C_1^{BM}(\mathbb{R})$$

Exercise: $\sum_{i \in \mathbb{Z}} \sigma_i$ generates $H_1^{BM}(\mathbb{R}) \cong \mathbb{Z}$.

Whitney Stratifications (CORRECTED): $n=\dim_{\mathbb{R}}(X)$

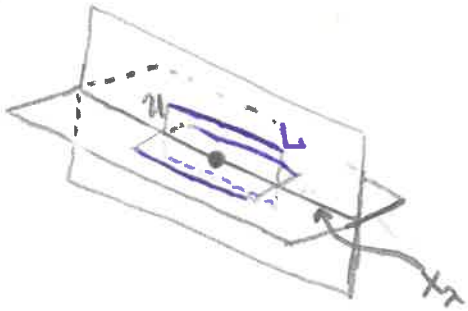
$X=\bigsqcup_\lambda X_\lambda$ is a stratification if $\overline{X_\lambda}$ is a union of X_μ for some set of μ.

This stratification is a Whitney stratification if

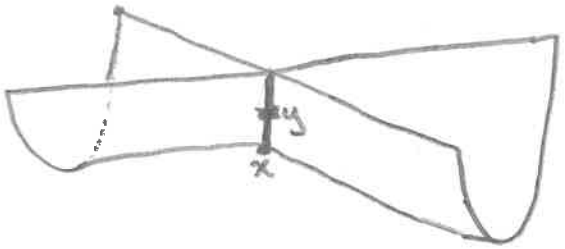
$\forall \lambda$, there is a stratified space $L=\bigsqcup_\mu L_\mu$ of $\dim(L)=n-\dim(X_\lambda)-1$ s.t. $\forall x\in X_\lambda$, $\exists$ neighborhood $\mathcal{U}\ni x$ s.t. $\phi: \mathcal{U} \overset{\text{homeo.}}{\approx} \mathbb{R}^{\dim(X_\lambda)} \times \operatorname{cone}(L)$, where

$$\operatorname{cone}(L)=L\times[0,1)/L\times\{0\}$$

s.t. $\phi^{-1}(L_\mu)=X_\mu\cap\mathcal{U}$ & $\phi|_{\mathcal{U}\cap X_\lambda}: \mathcal{U}\cap X_\lambda \xrightarrow{\sim} \mathbb{R}^{\dim(X_\lambda)}\times\{0\}$

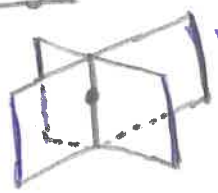


Ex: Whitney Umbrella: $\{x^2=zy^2\}\subseteq\mathbb{C}^3$



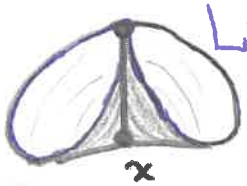
At y:

$y\in U$:



$\approx \mathbb{R}\times\operatorname{cone}(L)$

At x:



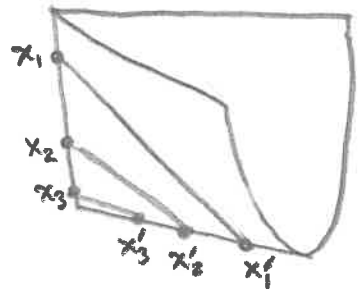
$L=S^1\vee S^1=(S^1\vee S^1\setminus\{pt\})\sqcup\{pt\}$ (for real picture)

These are equivalent to the Whitney conditions:

(a) If $x_i \longrightarrow x$ for $x_i \in X_\lambda$ & $x \in X_\mu$, then $T_x X_\mu \subseteq \lim_{i \to \infty} T_{x_i} X_\lambda$ if this limit exists

(b) If $x_i \in X_\lambda$ & $x_i' \in X_\mu$ both go to $x \in X_\lambda$, then $\lim_{i \to \infty} [\text{Secant line } (x_i, x_i')] \subseteq \lim_{i \to \infty} T_{x_i} X_\lambda$ if limits exist.

Ex: for Whitney umbrella, can take



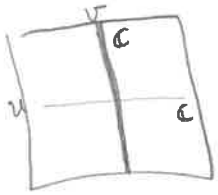
limit of secant lines is horizontal, but $\lim_{i \to \infty} T_{x_i} X_\lambda$ is vertical line only

So we need extra strata.

(Complex) Link of Whitney Umbrella:

$\pi: \mathbb{C}^2 \to X$
$(u,v) \longmapsto (uv, v, u^2)$

compatible w/ contraction $\lambda \cdot (u,v) = (\lambda u, \lambda^2 v)$ & $\lambda \cdot (x,y,z) = (\lambda^3 x, \lambda^2 y, \lambda^2 z)$



$\longrightarrow$ glues $\pm u$ coords

Link at $(0,0)$ is $X \setminus \{0\} / G_n \cong \mathbb{C}^2 \setminus \{0\} / \big((u,0) = (-u,0) \ \forall u,\ \lambda(\lambda u, v) = (u,v) \ \forall \lambda \forall u,v\big)$

$\cong S^3 / \big((u,0) = (-u,0)$ on an unknotted $S^1 \subset S^3\big)$

Links of Isolated Singularities:

Thm (Milnor fibers). $f: \mathbb{C}^{n+1} \to \mathbb{C}$ s.t. $f(0) = 0$, f: holomorphic

Let $B_\varepsilon = \{z \in \mathbb{C}^{n+1} \mid \|z\| \le \varepsilon\}$ be a small ball @ $0 \in \mathbb{C}^{n+1}$.

Then $f^{-1}(0) \cap B_\varepsilon \underset{\text{homeo.}}{\approx} \text{cone}(L)$ for $L \underset{\text{homotopic}}{\approx} \bigvee_\mu S^n$.

for $\mu = \dim_{\mathbb{C}} \dfrac{\mathbb{C}[\![x_0, \dots, x_n]\!]}{\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)}$

Ex: $f: \mathbb{C}^2 \to \mathbb{C}$, $(x,y) \longmapsto y^2 - x^3$ has $\mu = \dim_{\mathbb{C}} \dfrac{\mathbb{C}[\![x,y]\!]}{(2y, -3x^2)} = \dim_{\mathbb{C}} \text{Span}\{1, x\} = 2$

$L \approx S^1 \vee S^1$

Complex Link of $X = \{y^2 = x^3\} \subseteq \mathbb{C}^2$:

$\pi: \mathbb{C} \to X$, $z \longmapsto (z^2, z^3)$

Write $z = re^{i\theta}$ & fix r.

$\Rightarrow (r^2 e^{2i\theta}, r^3 e^{3i\theta})$ traverses 1 circle 3 x's & the other twice.

$\Rightarrow$ torus knot in S^3.

LECTURE 2:

Intersection Homology: X: singular $\mathbb{C}$-variety.

0: Sings. are interesting. e.g. $x^n = y^m$

I: Whitney Stratifications

$X = \coprod_\lambda X_\lambda$ stratification (ie. X_λ connd, locally clsd, smooth, $\overline{X_\lambda} = \bigcup_\mu X_\mu$)

We ask X_λ is "equisingular", i.e. $\forall x \in X_\lambda$, $\exists$ nghd $U \ni x$ s.t. $(X_\lambda \cap U) \times V \cong U$ (id on $X_\lambda \cap U$) ("local normal triviality")

Ex: Whitney umbrella $\{x^2 = zy^2\} \subseteq \mathbb{C}^3$.

$\mathbb{C}^2 \to$  $(u,v) \mapsto (uv, v, u^2)$

$X_{reg} \cup X_{sing}$ not good enough!

$X_{reg} \cup (X_{sing} \setminus 0) \cup \{0\}$

$y^2 + x^3 - t^2 x^2 = 0$: 

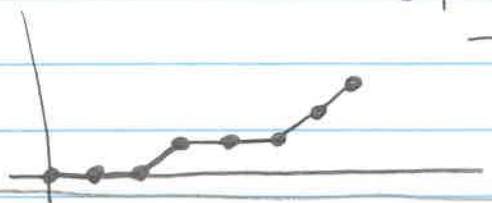
Whitney Condition: $X_\mu \subset \overline{X_\lambda} \Rightarrow \forall a_i \in X_\lambda$, $b_i \in X_\mu$ converging to $c \in X_\mu \to \lim_{i\to\infty}$ (secants $a_i - b_i$) $\in \lim_{i\to\infty} T_{a_i} X$

Intersection Homology: to recover P.D. / intersection thy, need to restrict what cycles we deal with.

Def: A perversity is a fxn $\bar{p} : [0, \dim X] \cap \mathbb{Z} \to \mathbb{N}$ (real dimension) s.t. $p_\alpha := \bar{p}(\alpha)$ satisfies $\begin{cases} p_0 = p_1 = p_2 = 0 \\ p_\alpha \le p_{\alpha+1} \le p_\alpha + 1, \ \alpha \ge 2 \end{cases}$

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e.g. 

 } slopes 0 or 1 everywhere.

e.g. upper middle perversity: $(0,0,0,1,1,2,2,\dots,\frac{n}{2}-1)$ ★ DEFAULT

lower middle: $(0,0,0,0,1,1,2,2,\dots,\frac{n}{2}-1,\frac{n}{2}-1)$

$X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$ $(\dim X_i \le i)$

An i-chain $\xi \in C_i(X)$ is $\bar{p}$-allowed if $\dim(|\xi| \cap X_{n-\alpha}) \le i - \alpha + p_\alpha$

Note: $\xi \pitchfork X_{n-\alpha} \iff \dim(|\xi| \cap X_{n-\alpha}) = i - \alpha$

$$IC_i^{\bar{p}}(X) = \left\{ \xi \in C_i(X) \;\middle|\; \begin{array}{l} \dim(|\xi| \cap X_{n-\alpha}) \le i - \alpha + p_\alpha \\ \dim(|\partial\xi| \cap X_{n-\alpha}) \le i - 1 - \alpha + p_\alpha \end{array} \ \forall \alpha \ge 2 \right\}$$

$$IH_i^{\bar{p}}(X; \mathbb{Z}) = H_i\left(IC_\bullet^{\bar{p}}(X)\right)$$