Splitting Criteria for Perverse Sheaves:

$X = X_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_0$, $Z_i = X_i - X_{i-1}$,

$U_i = X - X_i = \coprod_{j \geq i} Z_j$. Suppose $X = U_{\ell+1} \sqcup Z_\ell$

$$U = U_{\ell+1} \xrightarrow{j} X \xleftarrow{i} Z = Z_\ell$$

$$F \in Perv(X) \longrightarrow \tau_{\leq -\ell} F^\bullet \xrightarrow{\sim} F^\bullet$$

$$\& \ \tau_{\leq -\ell-1}(j^* F^\bullet) \xrightarrow{\sim} j^* F^\bullet$$

$$\Rightarrow \quad F \xrightarrow{t} Rj_* j^* F^\bullet \quad \text{(} \tau_{\leq -\ell}(Rj_* j^* F^\bullet) \text{ above)}$$

$$c : F \longrightarrow \tau_{\geq -\ell} F^\bullet \cong \mathcal{H}^{-\ell}(F^\bullet)[\ell]$$

supp'd in Z. so is a local system on Z.

$j^* F \in Perv(U)$ & Deligne's construction $\Rightarrow$

$$j_{!*} j^* F \cong \tau_{\leq -\ell-1} Rj_* j^* F$$

Thm: Assume $(\star)$ $\dim_{\mathbb{Q}} \mathcal{H}^{-\ell}(i_! i^! F^\bullet)_z = \dim_{\mathbb{Q}} \mathcal{H}^{-\ell}(i_* i^* F^\bullet)_z$ for some $z \in Z$. TFAE:

i) $\mathcal{H}^{-\ell}(i_! i^! F) \longrightarrow \mathcal{H}^{-\ell}(F)$ is isom.

ii) t lifts to $\tilde{t} : F \longrightarrow \tau_{\leq -\ell-1} Rj_* j^* F$ and $j_{!*} j^* F$

$(\tilde{t}, c) : F \longrightarrow j_{!*} j^* F \oplus \mathcal{H}^{-\ell}(F^\bullet)[\ell]$ is iso. of perv. sheaves.

not needed...

▷ gives a criteria for the filtr^n by artinian shvs to split.

e.g. self-dual $\Rightarrow (\star)$, so splitting holds inductively.

Pf: $i^* F \in {}^pD^{\leq 0}(Z)$ & $i^! F \in {}^pD^{\geq 0}(Z)$.

Z: sm. of dim $\ell \Rightarrow \mathcal{H}^k(i^* F) = 0 \ \forall \ k > -\ell$ & $\mathcal{H}^k(i^! F) = 0 \ \forall k < -\ell$.

$A := \mathcal{H}^{-\ell}(i^! F)[\ell]$; $B = \mathcal{H}^{-\ell}(i^* F)[\ell] \in Perv(Z)$

$$\rightsquigarrow \quad i_* A \xrightarrow{a} F \xrightarrow{c} i_* B$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\tau_{\leq -\ell} i_* i^! F \qquad\qquad \tau_{\geq -\ell} i_* i^* F$$

Recall: $i_* A$ = max'l sub. of F & $i_* B$ = max'l quot. of F.

(i)$\Rightarrow$(ii): $\mathcal{H}^{-\ell}(F)[\ell] \cong i_* \mathcal{H}^{-\ell}(i^* F)[\ell] = i_* B$ bc $\mathcal{H}^{-\ell}(F)$ is supp'd on Z.

(i) says that $i_* A \longrightarrow i_* B$ is isomorphism.

$\Rightarrow$ c is a split epimorphism. & a is split monomorphism.

(e.g. c splits via $a \circ (c \circ a)^{-1}$.) $\Rightarrow$ $F \cong \mathrm{Ker}(c) \oplus i_* B$

$\Rightarrow$ $\mathrm{Ker}(c) \cong j_{!*}(j^* F)$.

↑ no subs or quots. on Z.

ON BACK ⟶

(ii) $\Rightarrow$ (i): $\mathcal{H}^0(i^! j_{!*} j^* F) = 0$ so $\mathcal{H}^{-\ell}(i^! j_{!*} j^* F) = 0$

$B = \mathcal{H}^{-\ell}(F)[\ell]$ again has $\mathcal{H}^{-\ell}(i_* i^! F) \cong \mathcal{H}^{-\ell}(i_* B) \cong \mathcal{H}^{-\ell}(F)$

Thm (Artin's Vanishing Thm): X : affine complex n-dim'l variety. $\mathcal{F}$: constructible sheaf on X. Then $H^i(X,\mathcal{F}) = 0$ $\forall i > n$.

Rmk: $\mathcal{F}[n] \in {}^pD^{\leq 0}(X)$. $\exists$ version of above for $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$.

Pf Sketch:

$\exists\ \pi : X \longrightarrow \mathbb{A}^n$: finite, surjective by noether normalization

finiteness + proper base change $\Rightarrow R\pi_* \mathcal{F} \cong \underbrace{\pi_* \mathcal{F}}_{\text{constructible on } \mathbb{A}^n}$.

So suffices to prove on $\mathbb{A}^n$.

Generic Projection: $\exists$ non-$\emptyset$, open $U \subset \mathbb{A}^n$ s.t. $H^i(\mathbb{A}^n, j_! j^* \mathcal{F}) = 0$ $\forall i \neq n$.

Pf: Shrinking U, can assume $\mathcal{F}$ is a local system. & $Z = \mathbb{A}^n \setminus U$ is hypersurface. Choose generic proj^n $p : \mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$ s.t. $p|_Z$ is $\underbrace{\text{fin. + surj}^n}$.

Claim: $R^q p_* (j_! j^* \mathcal{F}) = 0$ for $q \neq 1$.

Assuming Claim, get generic projection by induction.

If we could do base change claim comes from following: fibers of p are $\mathbb{A}^1$, $Z_x = Z \cap \mathbb{A}^1$. $j_x : \underbrace{\mathbb{A}^1 - Z_x}_{U_x} \hookrightarrow \mathbb{A}^1$. Then $H^k(\mathbb{A}^1, j_{x!} \mathbb{Z}_{U_x}) = 0$ $\forall k \neq 1$.

e.g. $0 \to j_{x!} \underline{\mathbb{Q}}_{U_x} \longrightarrow \underline{\mathbb{Q}}_{\mathbb{A}^1} \longrightarrow \underline{\mathbb{Q}}_{Z_x} \to 0$

$\rightsquigarrow 0 \to \underbrace{H^0(\mathbb{A}^1, j_{x!} \underline{\mathbb{Q}}_{U_x})}_{\| \ 0} \to \mathbb{Q} \hookrightarrow \mathbb{Q}^{|Z_x|} \longrightarrow H^1(\mathbb{A}^1, j_{x!} \underline{\mathbb{Q}}_{U_x}) \to 0$

BUT: not proper, so need to be careful on base change!

p is an $\mathbb{A}^1$-bundle, so can cpt-fy to a $\mathbb{P}^1$-bundle

No problem bc Z does not go to ∞. //

Now, put $j : U \hookrightarrow X$ as above & $\mathcal{G} = j_! j^* \mathcal{F}$. $R^1 p_* \mathcal{G}$ is constructible sheaf on $\mathbb{A}^{n-1}$. Induct^n on n $\Rightarrow \exists V \underset{j_V}{\subset} \mathbb{A}^{n-1}$ non-$\emptyset$ open s.t. $H^k(\mathbb{A}^{n-1}, j_{V!}(R^1 p_* \mathcal{G}|_V))$ $\forall k \neq n-1$. $j' : U' = U \cap p^{-1}(V) \hookrightarrow \mathbb{A}^n$.

$R p_* j'_! (\mathcal{F}|_{U'}) \cong j_{V!}(R^1 p_* \mathcal{G}|_V)[-1] \rightsquigarrow H^k(\mathbb{A}^n, j'_!(\mathcal{F}|_{U'})) \cong H^{k-1}(\mathbb{A}^{n-1}, j_{V!}(R^1 p_* \mathcal{G}|_V))$

$= 0$ $\forall k-1 \neq n-1$

Now, we conclude w/an induction argument: $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ as in G.P.

$\to H^k(X, j_! j^* \mathcal{F}) \longrightarrow H^k(X, \mathcal{F}) \longrightarrow H^k(Z, i^* \mathcal{F}) \longrightarrow H^{k+1}(X, j_! j^* \mathcal{F}) \to \cdots$

Z affine of dim $\leq n-1$ $\Rightarrow H^k(Z, i^* \mathcal{F}) = 0$ $\forall k > n-1$. G.P. $\Rightarrow H^k(X, j_! j^* \mathcal{F}) = 0$ $\forall k > n$

$\Rightarrow H^k(X, \mathcal{F}) = 0$ $\forall k > n$ //

Thm. $f: X \to Y$ affine. Then Rf_* is right t-exact & $Rf_!$ is left t-exact. ▨

If f is qfinite, both are t-exact.

Pf: Since $\mathbb{D}_Y Rf_! = Rf_* \mathbb{D}_X$ & $\mathbb{D}_X({}^pD^{\geq 0}) = {}^pD^{\leq 0}$, it suffice to prove for one of Rf_* & $Rf_!$.

Lem: $K \in {}^pD^{\leq 0}(Y) \iff {}^pH^k(V,K) = 0 \ \forall V \subset Y$ affine open & $k > 0$.
($\&\ K \in {}^pD^{\geq 0}(Y) \iff H^k_c(V,K) = 0 \ \forall V \subset Y$ affine open & $k \geq 0$)

Pf: $\Rightarrow$: $E_2^{i,j} = {}^pH^i(V, {}^pH^j(K)) \Rightarrow H^{i+j}(V,K)$

$\operatorname{Supp} {}^pH^q(K)|_V \subset V$ is cl'd affine subvar.

Artin $\Rightarrow H^i(V, {}^pH^j(K)) = 0 \quad \forall\, i > -j$
(bc $\dim \operatorname{supp} {}^pH^j(K)|_V \leq -j$)

$\Rightarrow E_2^{i,j} = 0 \quad \forall\, i+j > 0$

& $H^k(V,K) = 0 \ \forall\, k > 0$.

$\Leftarrow$: **Fact:** If $F \in \mathrm{Perv}(X)$, then if $F \neq 0$, $\exists V \subset Y$ affine open s.t. $H^0(V,F) \neq 0$.

[$\exists$ simple sub $IC_S(\mathcal{L}) \subseteq F$ & can restrict to

$H^0(V, IC_S(\mathcal{I})) = H^d(V \cap S, \mathcal{I})$ ← so we get sections]

Suppose $H^k(V,K) = 0 \ \forall V \subset Y$ affine open. & $k > 0$. & assume $K \notin {}^pD^{\leq 0}(Y)$. Let $m > 0$ be largest s.t. ${}^pH^m(K) \neq 0$.

$${}^p\tau_{\leq m-1}K \to K \to \underbrace{{}^pH^m(K)}_{\cong P}[-m] \xrightarrow{[1]}$$

Choose $V \subset Y$ s.t. $H^0(V,P) \neq 0$.

$H^{m+1}(V, {}^p\tau_{\leq m-1}K) = 0$ bc $A \in {}^pD^{\leq (m-1)} \overset{\text{affine criteria}}{\iff} H^k(V, A[m-1]) = 0 \ \forall V \subset Y$ affine open $\forall k > 0$
$H^k(V,A) = 0 \ \forall k > m-1$

les $\Rightarrow H^m(V,K) \twoheadrightarrow H^0(V,P) \to 0 \Rightarrow H^m(V,K) \neq 0$ ↯ //

Rf_* is right t-exact: $K \in {}^pD^{\leq 0}(X)$. Suffices to show $H^k(V, Rf_*K) = 0$ $\forall k > 0$ $\forall V \subset Y$ affine open. $R\Gamma(V, Rf_*K) \cong R\Gamma(\underbrace{f^{-1}(V)}_{\text{affine}}, K) \overset{\text{Artin}}{=} 0 \ \forall k > 0$

$\overset{\text{affine criteria}}{\Rightarrow} Rf_*K \in {}^pD^{\leq 0}(Y)$. //

→

If f : qfinite, then f preserves dims of subvarieties & so preserves supp/cosupp conditions. //

Exs: ▷ $f=j: U \hookrightarrow X$ open embedding. Then R_{j*} & $R_{j!}$ are t-exact

▷ $f: X \to pt$ when X: affine

↳ Cor: X: complex, affine variety; $\mathcal{F} \in Perv(X)$. Then $\mathbb{H}^i(X;\mathcal{F}^\bullet)=0$ $\forall i>0$ & $\mathbb{H}^i_c(X,\mathcal{F}^\bullet)=0$ $\forall i<0$.

[bc $Rf_*\mathcal{F}^\bullet \in {}^pD^{\leq 0}(pt) = D^{\leq 0}(pt)$ & $Rf_!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(pt) = D^{\geq 0}(pt)$.
$\mathbb{H}^i(X,\mathcal{F}) = H^i(Rf_*\mathcal{F})$ & $\mathbb{H}^i_c(X,\mathcal{F}) = H^i(Rf_!\mathcal{F})$.]

Cor: X: lci, affine, $\dim_{\mathbb{C}} X = n$. Then
$H^i(X,\mathbb{Q})=0$ $\forall i>n$ & $H^i_c(X,\mathbb{Q})=0$ $\forall i<n$.

Pf: X: lci $\Rightarrow \underline{\mathbb{Q}}_X[n] \in Perv(X)$. Then apply previous Cor!

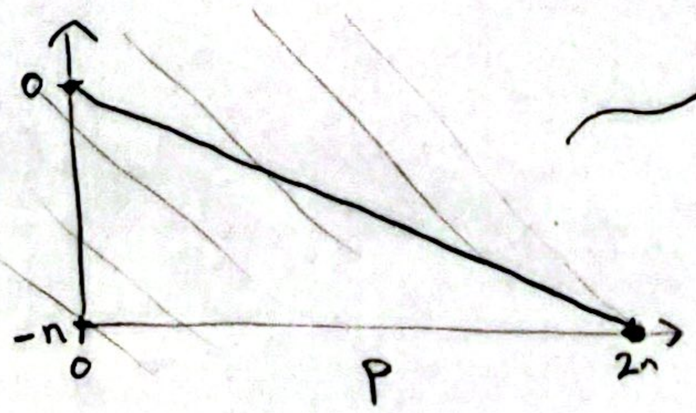
for X not affine, have something weaker:

Prop: X: pure, n-dim'l, $\mathcal{F} \in Perv(X)$. Then
$\mathbb{H}^i(X,\mathcal{F}) = \mathbb{H}^i_c(X,\mathcal{F}) = 0$ for $i \notin [-n,n]$.

Pf: $\mathcal{F} \in Perv(X)$, so $\mathcal{H}^i(\mathcal{F}^\bullet)=0$ $\forall i \notin [-n,0]$.
Also have hypercoh. spectral sequence:

$$E_2^{p,q} = H^p(X; \underbrace{\mathcal{H}^q(\mathcal{F}^\bullet)}_{\text{constructible of support dim} \leq n-q}) \Longrightarrow \mathbb{H}^{p+q}(X;\mathcal{F}^\bullet)$$

Artin's vanishing $\Rightarrow$ $H^p(X;\mathcal{H}^q(\mathcal{F}))=0$ for $p>-2q$



→ contributions are $\neq 0$ for $[-n,n]$ terms.