Intermediate Extension [4th definition of IC_X]

Def: $F^\bullet$ is an extension of $G^\bullet \in D^b_c(U)$ if $j^*F^\bullet \cong G^\bullet$.

$$\rightsquigarrow \quad j_! G \longrightarrow F \longrightarrow Rj_* G$$

"smallest extension" — "largest extension" — from adjunction

$$\Downarrow {}^pH^0 \text{ for } F^\bullet \in Perv(X)$$

$$P_{j_!} G \longrightarrow F \longrightarrow P_{j_*} G$$

Since $Perv(X)$ is an abelian category, we could construct $F^\bullet$ as an image.

Def: The intermediate extn $j_{!*} G^\bullet$ is the image of ${}^p j_! G \longrightarrow {}^p j_* G$ in $Perv(X)$.

① ★ [**Prop.** $j_{!*} G^\bullet$ is the unique perverse sheaf s.t. $i^*(j_{!*} G^\bullet) \in {}^pD^{\leq -1}(Z)$ and $i^!(j_{!*} G^\bullet) \in {}^pD^{\geq 1}(Z)$. (ie. $j_{!*} G$ has no perverse sub-objects or subquotients supp'd on Z)]

(Return to this!)

② **Prop.** Suppose $G^\bullet \in Perv(U)$, then $j_{!*} G^\bullet$ is unique, $F^\bullet$ s.t. $\forall X_\lambda \subset X - U$

① $\mathcal{H}^j(i_\lambda^* F^\bullet) = 0 \quad \forall j \geq -\dim_{\mathbb{C}} X_\lambda$

② $\mathcal{H}^j(i_\lambda^! F^\bullet) = 0 \quad \forall j \leq -\dim_{\mathbb{C}} X_\lambda$

Pf: Use Prop ★:

$$i^* j_{!*} F^\bullet \in {}^pD^{\leq -1}(Z) \iff \mathcal{H}^k(i_\lambda^*(j^* F^\bullet[1])) = 0 \quad \forall k > -\dim X_\lambda$$

$$\| \quad \mathcal{H}^{k+1}(i_\lambda^* F^\bullet)$$

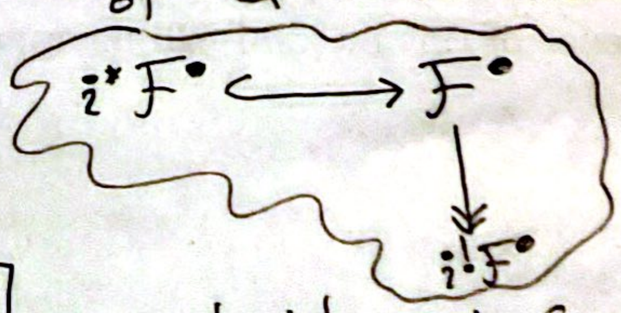
$$\iff \mathcal{H}^j(i_\lambda^* F^\bullet) \quad \forall j \geq -\dim X_\lambda$$

$$i^! F^\bullet \in {}^pD^{\geq 1}(Z) \iff \mathcal{H}^k(i_\lambda^!(i^! F^\bullet[-1])) = 0 \quad \forall k < -\dim X_\lambda$$

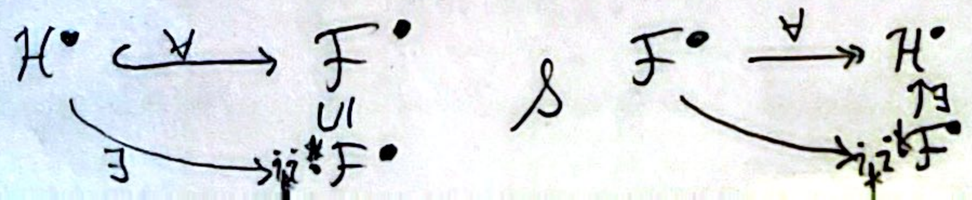
$$\| \quad \mathcal{H}^{k-1}(i_\lambda^! F^\bullet)$$

$$\iff \mathcal{H}^j(i_\lambda^! F^\bullet) \quad \forall j \leq -\dim X_\lambda \quad //$$

Pf of ★ Prop: Note any extension $F^\bullet$ of $G^\bullet$ in $Perv(X) = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ has $i^* F^\bullet \in {}^pD^{\leq 0}(Z)$ & $i^! F^\bullet \in {}^pD^{\geq 0}$.



If $\mathcal{H}^\bullet$ is supp'd on Z,



&

] no nontrivial subs & quots $\iff$ $i^* F^\bullet \in {}^pD^{\leq -1}(Z)$ & $i^! F^\bullet \in {}^pD^{\geq 1}(Z)$

i.e. $\mathrm{Hom}_X(F^\bullet, i_*Q^\bullet) = \mathrm{Hom}_Z(i^*F^\bullet, Q^\bullet) = \mathrm{Hom}_Z({}^pH^0(i^*F^\bullet), Q^\bullet)$

($i^*F^\bullet \in {}^pD^{\leq 0}(Z)$, $Q^\bullet \in \mathrm{Perv}(Z)$)

$\Rightarrow$ max'l quotient of $F^\bullet$ supp'd on Z is $i_*\,{}^pH^0(i^*F^\bullet)$

& $\mathrm{Hom}_X(i_*Q^\bullet, F^\bullet) = \mathrm{Hom}_Z(Q^\bullet, i^!F^\bullet) = \mathrm{Hom}_Z(Q^\bullet, {}^pH^0(i^!F^\bullet))$

($Q^\bullet \in \mathrm{Perv}(Z)$, $i^!F^\bullet \in {}^pD^{\geq 0}(Z)$)

$\Rightarrow$ max'l sub of $F^\bullet$ supp'd on Z is $i_*\,{}^pH^0(i^!F^\bullet)$

So: $\star$ condition $\iff$ $F^\bullet$ has no nontrivial subs. or quots. in Perv

$\Rightarrow$: $j_{!*}G$ has no subs or quots supported on Z.

$\parallel$ $\mathrm{Image}({}^pj_!G^\bullet \longrightarrow {}^pj_*G^\bullet)$

Pf: ▷ $\mathrm{Hom}_X({}^pj_!G, i_*Q) = \mathrm{Hom}_X({}^pH^0(Rj_!G), i_*Q) = \mathrm{Hom}_X(Rj_!G, i_*Q)$

$\cong \mathrm{Hom}_U(G, j^*i_*Q) = 0$

so no nontrivial quots.

▷ $\mathrm{Hom}_X(i_*Q, {}^pj_*G^\bullet) = \mathrm{Hom}_X(i_*Q, Rj_*G) = \mathrm{Hom}_U(j^*i_*Q, G) = 0$

so no nontrivial subs.

$\Leftarrow$: $F^\bullet$ extn w/ no subs & quots. on Z.

$\rightsquigarrow$ ${}^pj_!G \xrightarrow{a} F^\bullet \xrightarrow{b} {}^pj_*G$

Claim: a is surjective: $Q = \mathrm{coker}(a)$

Then $j^*Q = 0$ bc $G|_U = F^\bullet|_U$.

& Q = quot. of $F^\bullet$ $\Rightarrow Q = 0$

dually, b is injective

$\parallel$

Prop: $U_k := X - X_{2n-2k}$ & $\nu_k : U_k \hookrightarrow U_{k+1}$

(now labeling U's by dim)

Then $(j_k)_{!*}G^\bullet \cong {}^*\tau_{\leq k-n-1} R(j_k)_*G^\bullet$] inductive construction of $j_{!*}G$.

(${}^*\tau$: usual truncation)

Prop: $IC_X(\mathcal{L}) = j_{!*}(\mathcal{L}[n])$ for $\mathcal{L}$: local system on $U \subset X^{sm}$

Pf: (IA2) for $IC_X(\mathcal{L})$ says

$\mathcal{H}^{-k}(i_V^*IC_X(\mathcal{L})) = 0$ when $\dim(V) \geq k$ ← exactly condition on $j_{!*}(\mathcal{L}[n])$

- Dual condition follows from $\mathbb{D}_X(IC_X(\mathcal{L})) = IC_X(\mathcal{L}^\vee)$

Prop: If $\mathcal{G} \in Perv(U)$ is simple (in $Perv(U)$), $j_{!*}\mathcal{G}$ is simple (in $Perv(X)$).

Pf: $\mathcal{P}^\bullet \subset j_{!*}\mathcal{G}$ ∴ sub-object in $Perv(X)$

$\Rightarrow 0 \to \mathcal{P}^\bullet \xrightarrow{\alpha} j_{!*}\mathcal{G} \to \mathcal{Q}^\bullet \to 0$

Apply j^* $\Rightarrow$ by simplicity $j^*(\alpha)$ is iso. or 0.

$\Rightarrow$ either $\mathcal{P}^\bullet$ or $\mathcal{Q}^\bullet$ is supp'd on Z.

$\Rightarrow$ either $\mathcal{P}^\bullet = 0$ or $\mathcal{Q}^\bullet = 0$ //

~~A HARD result:~~ Not THAT hard result...

Thm (BBDG) 1998: ① $Perv(X)$ is an artinian category; ie. every perverse sheaf on X admits an increasing filtration whose quots. are simple.

① proved on back!

② The simple objects in $Perv(X)$ are $IC_{\bar{V}}(\mathcal{L})$, where $V = $ smooth irreducible alg. subvariety in X & $V \subset \bar{V}$ = clsr of V & $\mathcal{L}$ = local system on V.

② is easier than it looks: suffices to show simples on X of full support are IC's. Know on smooth part these are local systems by $Perv(X^{sm}) = Loc(X)[n]$. They must be intermediate extn. of these to be simple so they don't pick up extra subs or quots. on Z.

Prop. $\mathcal{G} \in Perv(U) \Rightarrow \mathbb{D}_X(j_{!*}\mathcal{G}) = j_{!*}\mathbb{D}_U\mathcal{G}$.

Pf: ${}^p j_! \mathcal{G} \twoheadrightarrow j_{!*}\mathcal{G} \hookrightarrow {}^p j_* \mathcal{G}$

$\mathbb{D}_X({}^p j_! \mathcal{G}) \leftarrow \mathbb{D}_X(j_{!*}\mathcal{G}) \leftarrow \mathbb{D}_X({}^p j_* \mathcal{G})$

$\|$ $\qquad\qquad\qquad

Proof of ①: Artinian prop. of $\mathrm{Perv}(X)$.

1) Suffices to show every $F \in \mathrm{Perv}(X)$ has a simple sub-object.
(+ finiteness arguments)

$S_1 \hookrightarrow F$ & $S_2 \hookrightarrow F/S_1$ ...

$\Rightarrow$ F_i = preimage of S_i in F.

$\mathbb{D}_X(F/F_i)$ gives filtration we claimed.

2) Induct on $\dim(X)$: $U \overset{j}{\subseteq} X$ open stratum.

$K = \mathrm{Image}\left({}^{p}j_! j^* F \longrightarrow F\right) \in \mathrm{Perv}(X)$.

↖ ext^n of j^*F.

If $j^*F = 0$, done by induction, so assume $j^*F = \mathcal{L}_U$ is non-0 local system on U. Choose $\mathcal{L}'_U \subset \mathcal{L}_U$: simple loc. system.

$\Rightarrow$ $K' := {}^{p}j_{!*}\mathcal{L}'_U[n]$ is simple sub-quot of F.

$${}^{p}j_!\mathcal{L}'_U[n] \longrightarrow {}^{p}j_!\mathcal{L}_U[n] \longrightarrow F$$
$$\searrow\ K'$$

Now, can assume F has no simple subs supported on Z. (else take those!)

$\Rightarrow$ $\mathrm{Image}\left({}^{p}j_!\mathcal{L}'_U[n] \longrightarrow F\right)$ has no non-zero subs on Z.

& bc ${}^{p}j_!\mathcal{L}'_U[n]$ has no quots on Z $\Rightarrow$ no non-zero quots.

$\Rightarrow$ $\mathrm{Image} = j_{!*}(\mathcal{L}'_U[n]) = K' \subset F$ is simple sub.