Ex: $IC_{\mathbb{C}}^{\bullet}(\mathbb{C}^*, \mathcal{L}_m) = j_* \mathcal{L}_m[1]$ (underived)

Why? Deligne's construction says:

$$IC_{\mathbb{C}}^{\bullet}(\mathbb{C}^*, \mathcal{L}_m) = \tau_{\leq -2} R j_* \mathcal{L}_m[1] = j_* \mathcal{L}_m[1].$$

Stalk @ 0: $\mathcal{H}^i(i_0^* j_* \mathcal{L}_m[1]) = \begin{cases} 0 & i \neq -2 \text{ or } m \neq 1 \\ \mathbb{Q} & i = -2 \ \& \ m = 1 \end{cases}$

$\Rightarrow (j_* \mathcal{L}_m)_0 \cong (\mathcal{L}_m)_x^{\pi_1}$ ← monodromy; ↑ general fiber

$j_* \mathcal{L}_x$	-2	-1
$\mathbb{C}^*$	V	0
0	V^T	V_T

VST

IC

	-2	-1
$\mathbb{C}^*$	V	0
0	V^T	0

Rmk: Same holds for any curve:

nodal curve: $IC^{\bullet}_{\{y^2 = x^3 - x^2\}} = \nu_* \underline{\mathbb{Q}}[1]$

ν

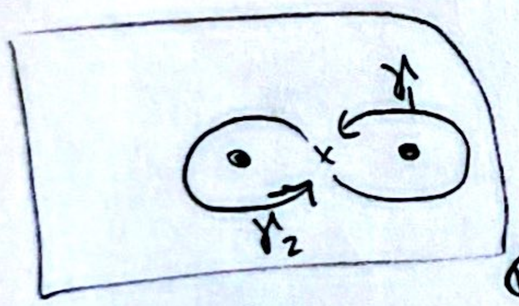
exact: $0 \to \mathbb{Q}[1] \longrightarrow \nu_* \underline{\mathbb{Q}}[1] \longrightarrow \mathbb{Q}_0[1] \to 0$

w/monodromy: $IC_x^{\bullet}(\mathcal{I}) = \nu_* IC_{\tilde{X}}(\nu^{-1}\mathcal{I}) = j_* \mathcal{I}[1]$

Actually holds whenever we have isolated singularities!

Let's compute for $\mathcal{I}$ = local system from H_1 of family of elliptic curves:

$\mathcal{I} \longleftrightarrow \begin{matrix} \gamma_1 \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ \gamma_2 \longmapsto \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \end{matrix}$



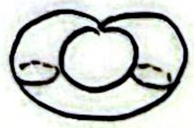
Isolated Whitney strat^n

$\Rightarrow IC(\mathcal{I}) = j_* \mathcal{I}[1]$

$(j_* \mathcal{I}[1])_1 \cong \mathcal{I}_x^{\gamma_1} = \ker\left(\mathbb{Q}^2 \xrightarrow{I_2 - \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}} \mathbb{Q}^2\right)$

$= \ker\left(\mathbb{Q}^2 \xrightarrow{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} \mathbb{Q}^2\right) = \mathbb{Q}$

Note: matches $H_1(\{y^2 = x^3(x-1)^2\}) = \mathbb{Q}$



$$\boxed{R^1 f_* \underline{\mathbb{Q}}_{\mathcal{E}} \cong IC_{\mathbb{A}^1}(\mathcal{I})[-1].}$$

Betti #'s: X: cx. alg. variety: $IC^{\bullet}_m = IC^{\bullet}_n$; $\dim_{\mathbb{C}} X = n$

Use $IC_X := IC^{\bullet}_m[-n]$ axis of symmetry is 0 now:

Poincaré Duality: $\mathbb{D}_X(IC_X) = IC_X$

(bc $\mathbb{D}_X(IC^{\bullet}_m[-n]) = \mathbb{D}_X(IC^{\bullet}_m)[n] = IC^{\bullet}_n[-2n][n] = IC_X$)

$IH^k(X;\mathbb{Q}) := \mathbb{H}^{k-n}(X; IC_X)$ & $IH^k_c(X;\mathbb{Q}) = \mathbb{H}^{k-n}_c(X; IC_X)$

Thm: X: complex alg. variety of $\dim_{\mathbb{C}} X = n$.

$U \xhookrightarrow{j} X \xhookleftarrow{i} Z = X \setminus U$

$\dim(Z) \le d$. Then

a) $IH^k(X) = IH^k(U)$ for $k < n-d$.

b) $IH^{n-d}(X) \hookrightarrow IH^{n-d}(U)$

c) $IH^{n+d}_c(X) \twoheadleftarrow IH^{n+d}_c(U)$

d) $IH^k_c(X) = IH^k_c(U)$ for $k > n+d$.

Rmk: Take X: projective & $d=0$ & $U = X^{sm}$:

$$IH^k(X) = \begin{cases} IH^k(U) & \text{for } k<n \\ IH^k_c(U) & \text{for } k>n+d \\ IH^n(X) \hookrightarrow IH^n(U) = H^n(U) & [\text{bc } n = \text{top dim}] \end{cases}$$

$IH^n_c(U) = H^n_c(X)$ $\twoheadrightarrow$ Im.

$\Rightarrow$ earlier result on IH of sm. variety with isolated sings.

Pf: $i_* i^! IC_X \to IC_X \to Rj_* j^! IC_X \xrightarrow{[1]} \cdots$

$\mathbb{H}^{k-n}(X;-)$ les:

$$\cdots \mathbb{H}^{k-n}(X; i_* i^! IC_X) \to \mathbb{H}^{k-n}(X; IC_X) \to \mathbb{H}^{k-n}(X; Rj_* j^! IC_X) \to \cdots$$

$$\mathbb{H}^{k-n}(Z, i^! IC_X) \qquad IH^k(X) \qquad IH^k(U)$$

(a) & (b) $\iff$ $\mathbb{H}^l(Z, i^! IC_X) = 0$ for $l \le -d$ (so. $k-n \le -d \leftrightarrow k \le n-d$)

Induct on d: Choose Whitney st. for Z & put $Z_1 = Z - Z^{sm}$ (= S). Shrink S s.t. S is union of Whitney strata of X.

$S \xhookrightarrow{j_S} Z \xhookleftarrow{i_1} Z_1$, $Z \xhookrightarrow{i} U$, $S \xrightarrow{i_S} U \xleftarrow{u} Z_1$

Costalk axiom $\Rightarrow$ $\mathcal{H}^l(i_S^! IC_X) = 0$ $\forall l \le -d$

Attaching $\Rightarrow$ les: $\cdots \to \mathbb{H}^l(Z_1, i_1^! F^{\bullet}) \to \mathbb{H}^l(Z, F^{\bullet}) \to \mathbb{H}^l(S, F^{\bullet}) \to \cdots$

$\forall l \le -(d-1)$ $0 = \mathbb{H}^l(Z_1, u^! IC_X)$ $\qquad$ $0 \ \forall l \le -d$

(ind. hyp.)

$\Rightarrow \quad \mathbb{H}^l(\bar{z}, F^\circ) = 0 \quad \forall l \leq -d$ ☑

Dual direction is done w/ $\mathbb{H}_c^{k-n}(X; -)$ of:

$$j_! j^* IC_X \longrightarrow IC_X \longrightarrow i_* i^* IC_X \xrightarrow{[1]} \cdots$$

& an identical (dual) argument

Ex: