Examples: • Standard t-structure:

$$D^{\geq 0} = \{ F^\bullet \in D^b_c(X) \mid \mathcal{H}^i(F^\bullet) = 0 \quad \forall i > 0 \}$$

$$D^{\leq 0} = \{ F^\bullet \in D^b_c(X) \mid \mathcal{H}^i(F^\bullet) = 0 \quad \forall i < 0 \}$$

$$\tau_{\leq 0}(F^\bullet) = [\cdots \to F^{-2} \to F^{-1} \to \ker(d^0) \to 0 \to \cdots]$$

$$\tau_{\geq 0}(F^\bullet) = [\cdots \to 0 \to \operatorname{coker}(d^{-1}) \to F^1 \to F^2 \to \cdots]$$

$$\mathcal{C} \cong Sh_c(X)$$

• Perverse t-structure:

(★)
$$^pD^{\leq 0} = \{ F^\bullet \in D^b_c(X) \mid \dim_{\mathbb{C}} \operatorname{supp}^{-j}(F^\bullet) \leq j \quad \forall j \in \mathbb{Z} \}$$

$$^pD^{\geq 0} = \{ F^\bullet \in D^b_c(X) \mid \dim_{\mathbb{C}} \operatorname{cosupp}^{j}(F^\bullet) \leq j \quad \forall j \in \mathbb{Z} \}$$

$$\left[\operatorname{supp}^j(F^\bullet) = \overline{\{ x \in X \mid \mathcal{H}^j(i_x^* F^\bullet) \neq 0 \}} \ldots \right]$$

$$Perv(X) = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}.$$

${}^p\tau_{\leq m}$ & ${}^p\tau_{\geq m}$ = "perverse truncation"

${}^p\mathcal{H}^i$ = perverse cohomology functor.

Rmk: UCT $\Longrightarrow$ Lem: If A = field, $\operatorname{cosupp}^j(F^\bullet) = \operatorname{supp}^{-j}(\mathbb{D}_X F^\bullet)$

Pf: $\mathcal{H}^j(i_x^! F^\bullet) = \mathcal{H}^j(i_x^! \mathbb{D}_X \mathbb{D}_X F^\bullet) = \mathcal{H}^j(\mathbb{D}_x i_x^* \mathbb{D}_X F^\bullet)$

$\overset{\text{UCT}}{\cong} \mathcal{H}^{-j}(i_x^* \mathbb{D}_X F^\bullet)^*$

$\uparrow \operatorname{supp}^{-j}(\mathbb{D}_X F)$ computes here

$\operatorname{cosupp}^j(F^\bullet)$ computes here.

(★) A: field

Cor: $\mathbb{D}_X({}^pD^{\leq 0}) = {}^pD^{\geq 0}$ & so in particular, $\mathbb{D}_X : Perv(X) \to Perv(X)$

Rmk: Also: ${}^p\tau_{\leq 0} \circ \mathbb{D}_X = \mathbb{D}_X \circ {}^p\tau_{\geq 0}$

${}^p\tau_{\geq 0} \circ \mathbb{D}_X = {}^p\tau_{\leq 0} \circ \mathbb{D}_X$

$$\mathbb{D}_X({}^p\mathcal{H}^j(F^\bullet)) = {}^p\mathcal{H}^{-j}(\mathbb{D}_X F^\bullet)$$

Ex: $IC_X \in Perv(X)$ [same for $IC_X(\mathcal{L})$]

Pf: Recall $IC_X[n]$ satisfies

IA2: $\dim_{\mathbb{R}} \operatorname{supp}^j(IC_X[n]) \leq 2n - p^{-1}(j+2n)$

$\dim_{\mathbb{R}} \operatorname{supp}^{j+n}(IC_X) \leq 2n - p^{-1}(j+2n) \overset{p=m}{=} 2n - 2j - 4n = -2(j+n)$

$\iff \dim_{\mathbb{C}} \operatorname{supp}^{-i}(IC_X) \leq i-1 \; \forall i$ ← Support Condition!

$m^{-1}(s) = 2s+2$ $\quad (m(r) = \lfloor \frac{r-2}{2} \rfloor)$, $r \geq 2$

~~Verdier dual of $IC_X[n]$ satisfies $D(IC_X[n])$~~

Similar for cosupport:

$\dim_{\mathbb{R}} \operatorname{cosupp}^j(IC_X[n]) \leq 2n - q^{-1}(-j) \overset{q=n}{=} 2n + 2j - 1$

$\dim_{\mathbb{R}} \operatorname{cosupp}^{2n+j}(IC_X)$

$\iff \dim_{\mathbb{C}} \operatorname{cosupp}^i(IC_X) \leq i-1 \; \forall i$ ← Cosupport condition!

Note: IC condition is slightly stronger ($i-1$ vs i).

Perverse Sheaves carry a def^n w/ Whitney strat^n:

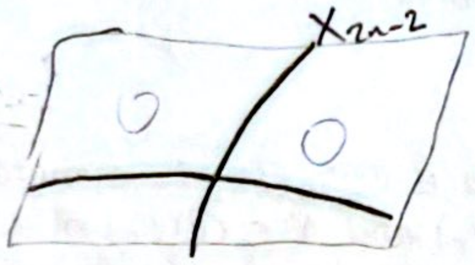
Thm: $\mathcal{X}: X = \bigsqcup_\lambda X_\lambda$: Whitney, $i_\lambda : X_\lambda \hookrightarrow X$. If $\mathcal{F}^\bullet$ is constr. wrt $\mathcal{X}$ then

$$\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X) \iff \mathcal{H}^j(i_\lambda^* \mathcal{F}^\bullet) = 0 \quad \forall \lambda \ \forall j > -\dim_{\mathbb{C}} X_\lambda$$

$$\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X) \iff \mathcal{H}^j(i_\lambda^! \mathcal{F}^\bullet) = 0 \quad \forall \lambda \ \forall j < -\dim_{\mathbb{C}} X_\lambda$$

Lem: $\mathcal{F}^\bullet \in \mathrm{Perv}(X) \implies \mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $i \notin [-\dim_{\mathbb{C}} X, 0]$

Pf:



$\mathcal{H}^{-\dim_{\mathbb{C}} X_\lambda}$ supported anywhere

$\mathcal{H}^{-\dim_{\mathbb{C}}(X_\lambda)+1}$ supported on X_{2n-2}

$\mathcal{H}^{-\dim_{\mathbb{C}}(X_\lambda)+2}$ supp'd on X_{2n-4}

$\vdots$ $\vdots$

$\mathcal{H}^0$ supported on X_0 //

(similar w/ costalks bdds opposite direction.)

Rmk: IC_X conditions are actually weaker than $\mathrm{Perv}(X)$, even w/ normalization condition.

e.g. $\underline{\mathbb{Q}}_X[1] \oplus i_* \mathbb{Q}$ is perverse.

$i : \{x\} \hookrightarrow X = \mathbb{A}^1$

Ex: $\mathcal{L}[n] \in \mathrm{Perv}(X)$ for X : smooth & $\mathcal{L}$: local system on X.

↑ not true in general for non-smooth X.

Interesting example: $X =$ complete intersection. Then $\mathcal{L}[n] \in \mathrm{Perv}(X)$ $\forall$ local system $\mathcal{L}$ on X.

Gluing t-structures:

Def: A functor $F: D_1 \longrightarrow D_2$ can be left / right t-exact

$${}^pF: \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \quad \text{by} \quad {}^tH^0 \circ F$$

X: complex alg. variety w/ $\mathcal{X}: X = X_{2n} \supset \cdots \supset X_0$ Whitney strat^n.

$Z \subset X$: union of strata & $U = X - Z$

$$U \xhookrightarrow{j} X \xhookleftarrow{i} Z$$

Facts about i, j:

- All $*$ & $!$ functors of i & j preserve constructibility.
- $j^* \circ i_* = 0$, $i^* \circ j_! = 0$, $i^! \circ Rj_* = 0$.

  bc $\mathrm{Hom}(\mathcal{G}, i^! Rj_* \mathcal{F}) = \mathrm{Hom}(i_* \mathcal{G}, Rj_* \mathcal{F})$
  $= \mathrm{Hom}(j^! \circ i_* \mathcal{G}, \mathcal{F}) = 0$
- Adjunctions: $i^* i_* \longrightarrow \mathrm{id} \longrightarrow i^! i_!$
  $j^* Rj_* \longrightarrow \mathrm{id} \longrightarrow j^! j_!$
  are isomorphisms.

Thm (gluing).

$${}^pD^{\leq 0}(X) = \left\{ \mathcal{F}^\bullet \in D^b_c(X) \;\middle|\; j^*\mathcal{F}^\bullet \in {}^pD^{\leq 0}(U),\ i^*\mathcal{F}^\bullet \in {}^pD^{\leq 0}(Z) \right\}$$

$${}^pD^{\geq 0}(X) = \left\{ \mathcal{F}^\bullet \in D^b_c(X) \;\middle|\; j^!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(U),\ i^!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(Z) \right\}$$

Pf: stalks/costalks are computed on U & Z:

$$\mathrm{supp}^j(\mathcal{F}^\bullet) = \mathrm{supp}^j(j^*\mathcal{F}^\bullet) \cup \mathrm{supp}^j(i^*\mathcal{F}^\bullet)$$

& $\mathrm{cosupp}^j(\mathcal{F}^\bullet) = \mathrm{supp}^j(j^!\mathcal{F}^\bullet) \cup \mathrm{cosupp}^j(i^!\mathcal{F}^\bullet)$ //

Cor: $Rj_!, i^*$ are right t-exact $({}^pD^{\geq 0} \longrightarrow {}^pD^{\geq 0})$

$j^* = j^!, i_* = i_!$ are t-exact $(\mathrm{Perv} \longrightarrow \mathrm{Perv})$

$Rj_*, i^!$ are left t-exact $({}^pD^{\leq 0} \longrightarrow {}^pD^{\leq 0})$

Cor: If $\bigcup_j \mathrm{Supp}^j(\mathcal{F}^\bullet) \subset Z$, then $\mathcal{F}^\bullet \in \mathrm{Perv}(X) \Leftrightarrow i^*\mathcal{F}^\bullet \in \mathrm{Perv}(Z)$.

Pf: We claim $i^*\mathcal{F}^\bullet \cong i^!\mathcal{F}^\bullet$: $\mathrm{Supp}(\mathcal{F}^\bullet) \subset Z \Rightarrow j^*\mathcal{F}^\bullet = 0$

$$\Rightarrow \mathcal{F}^\bullet \cong i_! i^! \mathcal{F}^\bullet \cong i_* i^* \mathcal{F}^\bullet \xRightarrow{i^* \circ} i^!\mathcal{F}^\bullet \cong i^*\mathcal{F}^\bullet$$

Then the Cor follows from t-exactness properties of previous cor.

Perverse Sheaves:

new conventions: $p = m$ is middle $\Rightarrow IC_X = IC^{\bullet}_m[-n]$ for $n = \dim_{\mathbb{C}}(X)$.

$$[\mathbb{D}_X[IC_X] = IC_X]$$

The category of perverse sheaves is a subcat.

$$Perv(X) \subseteq \underbrace{D^b_c(X)}_{=:D} \text{ - constructible cat. of } X.$$

Def: A t-structure on D is the data of $(D^{\leq 0}, D^{\geq 0})$ s.t.

$D^{\leq n} = D^{\leq 0}[n]$ & $D^{\geq n} = D^{\geq 0}[-n]$ satisfy:

① $\mathrm{Hom}_D(P, R) = 0 \quad \forall P \in D^{\leq 0}$ & $Q \in D^{\geq 1}$.

② $D^{\leq 0} \subseteq D^{\leq 1}$ & $D^{\geq 1} \subseteq D^{\geq 0}$.

③ $\forall P \in Ob(D)$, $\exists \Delta$ in D.

$$P' \longrightarrow P \longrightarrow P'' \xrightarrow{[1]} \quad \text{s.t. } P' \in D^{\leq 0} \text{ & } P'' \in D^{\geq 1}.$$

In this case we call $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$ the heart of the t-structure above.

Def. A t-structure is non-deg. if $\bigcap_n D^{\leq n} = \bigcap_n D^{\geq n} = \{0\}$.

We get truncation functors $\tau_{\leq n}$ & $\tau_{\geq n}$ so that

$$\tau_{\leq 0} P \longrightarrow P \longrightarrow \tau_{\geq 1} P \xrightarrow{[1]}$$

Prop: $\mathcal{C}$ is an abelian category which is stable by extensions.

Pf: $\mathcal{C} = \tau_{\leq 0}\tau_{\geq 0}(D)$.

For $f: A \to B$ in $\mathcal{C}$,

$A \to B \to C = \mathrm{cone}(f) \xrightarrow{[1]}$

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$\Rightarrow C \in D^{\geq -1} \cap D^{\leq 0}$

les a t-cohomology $\Rightarrow 0 \to H^{-1}_t(C) \to A \to B \to H^0_t(C) \to 0$

→ **Def.** ${}^tH^0 = \tau_{\leq 0}\tau_{\geq 0} = \tau_{\geq 0}\tau_{\leq 0}$ & ${}^tH^i = {}^tH^0 \circ [i]$

$:= \tau_{\geq i}\tau_{\leq i}[i]$

Have a long exact seq. from ${}^tH^i$ ...

Proof that heart is abelian:

$f: A \to B$ in $\mathcal{C}$. $\rightsquigarrow$ complete to exact Δ:

$$A \xrightarrow{f} B \to C \xrightarrow{[1]}$$

[redacted] $B \to C \to A[1]$

$D^{\leq 0} \cap D^{\geq 0}$ $\qquad$ $D^{\geq -1} \cap D^{\leq -1}$

$D^{\geq -1}, D^{\leq 0}$ clsd under extns $\Rightarrow C \in D^{\geq -1} \cap D^{\leq 0}$

Put $K = \tau_{\leq -1} C[-1]$ & $Q = \tau_{\geq 0} C$

$K \in D^{\leq -1}$, $Q \in D^{\geq 0}$

$\rightsquigarrow$ $K[1] \to C \to Q \xrightarrow{[1]}$

$\phi : K \to C[-1] \to A$ has $f\phi = 0$ (bc $f\phi =$ $K \to C[-1] \to A \to B$, distinguished)

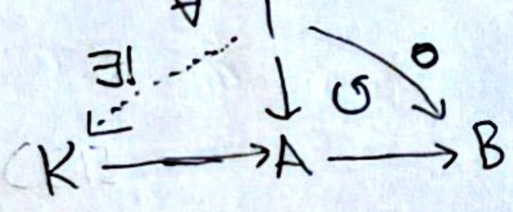
$T \in \mathcal{C}$ any test object $\Rightarrow$ from usual les:

$$\mathrm{Hom}(T, B[-1]) \to \mathrm{Hom}(T, C[-1]) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B)$$

$\mathrm{Hom}(T, B[-1]) = 0$ ← no maps between $D^{\leq 0}$ & $D^{\geq 1}$

$\Rightarrow \mathrm{Hom}(T, C[-1]) \cong \ker\left(\mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B)\right)$

$= \mathrm{Hom}(T, K)$.



$\Rightarrow K = \ker(f)$

Dual argument $\Rightarrow Q = \mathrm{coker}(f)$

Pf of les on ${}^{t}\mathcal{H}^{i}$: Similar to usual les on cohomology (EXERCISE!)