① Same as ②. //

Cor: ∃ nondeg. pairing

$$I^{BM}H_i^p(X,A) \otimes I H_{n-i}^q(X,A) \longrightarrow A .$$

Pf:
$$I^{BM}H_i^p(X,A) \cong \mathbb{H}^{-i}(X, IC_p^\bullet) \cong \mathbb{H}^{-i}(X, D_X IC_q^\bullet[n])$$
$$\cong \mathbb{H}^{n-i}(X, D_X IC_q^\bullet) \overset{UCT}{\cong} \mathbb{H}_c^{-n+i}(X, IC_q^\bullet)^* \cong I H_{n-i}^q(X,A)^* \;//$$

Small adaptation gives:

$$D_X(IC_q^\bullet(\mathcal{L}^\vee)[n]) \cong IC_p^\bullet(\mathcal{L})$$

& so also get:

$$I^{BM}H_i^p(X;\mathcal{L}^\vee) \cong I H_{n-i}^q(X,\mathcal{L})$$

Q: All of this still depends on a Whitney stratif^n.
Is $IC_p^\bullet$ independent of choice?

Thm: $\mathcal{X}_1, \mathcal{X}_2$: 2 Whitney strat^n's on X. Then

$$IC_{p,\mathcal{X}_1}^\bullet \underset{\text{qiso.}}{\simeq} IC_{p,\mathcal{X}_2}^\bullet .$$

Pf Sketch: ▷ Find equivaleat, intrinsic axioms that are equiv. to ours & show they characterizes $IC_p^\bullet$ indep. of strat^n.

Intrinsic Axioms: $\mathcal{A}^\bullet \in D(X)$:

$$\operatorname{supp}^j(\mathcal{A}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^* \mathcal{A}^\bullet) \neq 0\}}$$
$$\operatorname{cosupp}^j(\mathcal{A}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^! \mathcal{A}^\bullet) \neq 0\}}$$
$$p^{-1}(j) = \begin{cases} \min\{k : p(k) \geq j\} & , j \leq p(n) \\ \infty & , j > p(n) \end{cases}$$

⓪ (normal^n) $\exists U \subset X$ open s.t. $\mathcal{A}^\bullet|_U \cong \omega_U = \underline{A}_U[n]$ } same
① (lower bd) $\mathcal{H}^j(\mathcal{A}^\bullet) = 0 \quad \forall j < -n$
②′ (support) $\dim \operatorname{supp}^j(\mathcal{A}^\bullet) \leq n - p^{-1}(j+n) \quad \forall j > -n$
③ (cosupport) $\dim \operatorname{cosupp}^j(\mathcal{A}^\bullet) \leq n - q^{-1}(j) \quad \forall j < 0$. for q: compl. perversity

on back →

▷ We have already shown $IC^{\bullet}_{p,\mathcal{X}}$ satisfies these intrinsic axioms.

▷ In opposite direction, choose a strat^{n} $\mathcal{X}: X_n \supset X_{n-1} \supset \cdots \supset X_0$ s.t. stalk & costalk coh. are locally constant on each strata.

▷ Let $S_k = X_{n-k} - X_{n-k-1}$ codim k. $x \in S_k$:

if $H^r(i_x^* A^\bullet) \neq 0$, the support bound $\Rightarrow$ $-k \leq n - p^{-1}(r) \iff r \leq p(k) - n$

$$\Rightarrow H^r(i_x^* A^\bullet) = 0 \quad \forall r > p_k - n$$

& cosupport $\longrightarrow$ $H^r(i_x^! A^\bullet) = 0 \quad \forall r < -q_k = p_k - n + 2$

$$i^! A^\bullet \longrightarrow i^* A^\bullet \longrightarrow i^* Rj_* j^* A^\bullet \xrightarrow{[1]} \cdots$$

($i^! A^\bullet$: in degs $r \geq -q_k = p_k - n + 2$; $i^* A^\bullet$: $r \leq p_k - n$)

$\Rightarrow H^r(i^* A^\bullet) \longrightarrow H^r(i^* Rj_* j^* A^\bullet)$ is isom. $\forall r \leq p_k - n$.

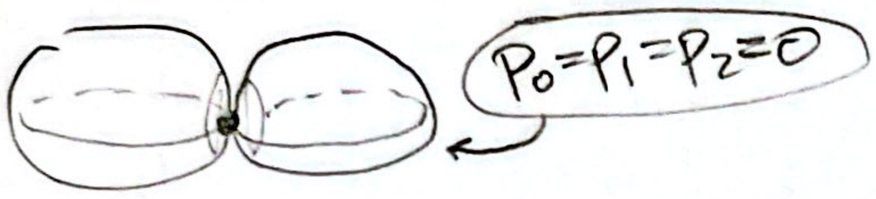
and $H^r(i^* A^\bullet) = 0 \quad \forall r > p_k - n$

$\Rightarrow i^* A^\bullet \cong \tau_{\leq p_k - n}(i^* Rj_* j^* A^\bullet)$ [Deligne's construction.]

Example:

$\mathbb{P}^1 \cup \mathbb{P}^1 \subset \mathbb{P}^2 \Rightarrow$ $j: \mathbb{A}^1 \sqcup \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \cup \mathbb{P}^1$

$$IC^\bullet = \tau_{\leq p_2 - n} Rj_* \underline{A}_{U_2}[2] = \tau_{\leq(-2)} Rj_* \underline{A}_{U_2}[2].$$



$$\mathcal{H}^r\left(i_x^* Rj_* \underline{A}_{U_2}[2]\right) = \varinjlim_{V \ni x} H^{r+2}(V \setminus \{x\}, A)$$

($V \setminus \{x\} \sim S^1 \sqcup S^1$)

$$= \begin{cases} A^2 & , r = -2, -1 \\ 0 & , \text{else} \end{cases}$$

$$\Rightarrow \mathcal{H}^r(i_x^* IC^\bullet) = \begin{cases} A^2 & , r = -2 \\ 0 & , \text{else} \end{cases}$$

In fact: $IC^\bullet \cong \nu_* \underline{A}_{\mathbb{P}^1 \sqcup \mathbb{P}^1}[2]$ ← computed via exact Δ: $\underline{A}_X \to \nu_* \underline{A}_{\tilde{X}} \to A_x$ (also computes ω_X.)

$\nu: \mathbb{P}^1 \sqcup \mathbb{P}^1 \to \mathbb{P}^1 \cup \mathbb{P}^1$ normal^n

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Rmks: (common confusion on shifts)

$H^i(\mathcal{F}^\bullet[j]) = H^{i+j}(\mathcal{F}^\bullet)$ moves coh. more negative $\Rightarrow$ i

($\mathcal{F}^\bullet[j]$ is shifted left/negative direction!)

Relation with normalizn:

Prop: $\nu: \tilde{X} \to X$: normalizn. Then $IC^\bullet_{X,p} \cong R\nu_* IC^\bullet_{\tilde{X},p}$

Pf: Check axioms: $\nu = \mathrm{Id}$: on smooth locus $\Rightarrow$ normalizn.

ν : proper $\Rightarrow$ by proper base change, $i_x^* R\nu_* IC_{\tilde{X}} \cong \bigoplus_{y \in \nu^{-1}(x)} i_y^* IC_{\tilde{X}}$

$$\Rightarrow \{x : H^r(i_x^* R\nu_* IC_{\tilde{X}}) \neq 0\} = \nu\left(\{y : H^r(i_y^* R\nu_* IC(\tilde{X}) \neq 0\}\right)$$

preserves dims $\Rightarrow$ support condition

Proper base change for costalks gives same for co-support.

Rmk: Works fine w/local system coeffs: $\mathcal{L}$: loc. sys. on $X^{sm} = U$.

$$\Rightarrow IC^\bullet_{X,p}(\mathcal{L}) \cong R\nu_* IC^\bullet_{\tilde{X},p}(\nu^* \mathcal{L}|_{\nu^{-1}(U)})$$

Ex: Rational Homology Mfds: (or $\mathbb{Q}$-mfd)

Def: X is a rational homology mfd. of dim m if

$$\forall x \in X,\quad H_i(X, X-x; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i=m \\ 0, & \text{else.} \end{cases}$$

Ex: If M is a smooth mfd & G: finite gp, $G \curvearrowright M$. Then

$$X = M/G \text{ is a } \mathbb{Q}\text{-mfd.}$$

Prop. X is a $\mathbb{Q}$-mfd $\iff$ the link of any $x \in X$ is a rational homology sphere, ie.

$$H_i(L_x, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = \dim L_x \\ 0, & \text{else.} \end{cases}$$

Pf: use excision to write

$$H_i(X, X-x, \mathbb{Q}) \cong H_i(U_x, U_x - x, \mathbb{Q}) \text{ for disting. nghd } U_x \text{ \& then use cone computation.} \quad //$$

Thm. ($p = m$ middle perversity.) X: cx alg. var. of dim n which is a $\mathbb{Q}$-mfd. Then $IC^{\bullet}_m \cong \mathbb{Q}_X[2n]$.

Pf: $IC_X := IC^{\bullet}_m[-n]$ (in degs $-n, \dots, n$ now)

▷ normaliz^n & bdd-ness are obvious

▷ $\mathcal{H}^j(\underline{\mathbb{Q}}_X[n])_x \cong \mathbb{H}^j(U_x; \underline{\mathbb{Q}}_X[n]) = H^{n+j}(U_x, \mathbb{Q}) = 0$ when $j+n > 0$ since U_x is contractible.

(so holds for $j \geq -n + \text{codim}$)

▷ $x \in X_{n-2k}$; $\dim(L_x) = 2k-1$ & L_x is $\mathbb{Q}$-homology sphere

(Σ even bc cx alg. variety.)

$$\forall x \in X_{n-2k},\quad \mathcal{H}^j(i_x^! \underline{\mathbb{Q}}_X[n]) \cong \mathbb{H}^j_c(U_x; \mathbb{Q}_X[n]) = H^{j+n}_c(U_x; \mathbb{Q})$$

$$\underset{\text{Kunneth}}{\cong} \bigoplus_{p+q=j+n} H^p_c(\mathbb{C}^{n-k}; \mathbb{Q}) \otimes H^q_c(cL_x, \mathbb{Q})$$

$$= \begin{cases} 0, & j \neq n \\ \mathbb{Q}, & j = n \end{cases} \implies \mathcal{H}^j(i_x^! \mathbb{Q}_X[n]) = 0$$

$\forall\, j \leq n - \text{codim} = n - 2k.$