The Intersection Cohomology Complex:

Goal: sheafify intersection homology

So far: if X is **smooth**:

$H^*(X, \mathbb{Q}) \rightsquigarrow \underline{\mathbb{Q}}_X$ s.t. $H^*(X, \underline{\mathbb{Q}}_X) \cong H^*(X, \mathbb{Q})$

Poincaré Duality $\rightsquigarrow$ Verdier duality $\mathbb{D}_X$

~~X: locally ... $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ Whitney ...~~

Recall: to compute $H^*(X, \underline{\mathbb{Q}}_X)$, we used a **resol^n** of $\underline{\mathbb{Q}}_X$:

Variant for homology: $C^\bullet \longrightarrow \underline{\mathbb{Q}}_X$

$C^{-i}(U) = C_i(U)$: i-chains on U

∂ = connecting map for $C_\bullet(U)$.

Generalize to IC: p = perversity fcn,

$IC_p^{-i}(U) := I^{BM}C_i^p(U) \rightsquigarrow IC_p^\bullet \in D^b(X)$

Recall: $\mathcal{F} \in Sh(X)$ is **soft** if $\forall$ cpt $K \subset X$,

$\Gamma(X, \mathcal{F}) \xrightarrow{res} \Gamma(K, \mathcal{F}) := \varinjlim_{U \supset K} \Gamma(U, \mathcal{F})$ is surjective

Lem: IC_p^{-i} is a soft sheaf (& so computes $Rf_*, Rf_!$)

$\Longrightarrow$ **Prop:** $I^{BM}H_i^p(X) = \mathbb{H}^{-i}(X, IC_p^\bullet) := R^{-i}\Gamma(IC_p^\bullet)$

$IH_i^p(X) = \mathbb{H}_c^{-i}(X, IC_p^\bullet) := R^{-i}\Gamma_c(IC_p^\bullet)$

In fact $IC_p^\bullet$ will not be enough simple objects for category of perverse sheaves later: even on **smooth** X, also have $C^\bullet(\mathcal{L})$ computing $H^i(X, \mathcal{L})$ for $\mathcal{L}$: local system on X

[Return to $H^*(X, \mathcal{L})$ here now...]

Define $IC_p^\bullet(\mathcal{L})$ similarly.

$I^{BM}H_i^p(X, \mathcal{L}) = \mathbb{H}^{-i}(X; IC_p^\bullet(\mathcal{L}))$ & $IH_i^p(X, \mathcal{L}) = \mathbb{H}_c^{-i}(X, IC_p^\bullet(\mathcal{L}))$

(Co)Homology with coefficients in a local system:

for $\sigma : \Delta^i \longrightarrow X$ & $\mathcal{L}$: local system $\Longrightarrow \sigma^*\mathcal{L}$: loc. sys. on Δ^i

Note Δ^i = contractible, so $\sigma^*\mathcal{L}$ = constant shf.

$\mathcal{L}_\sigma := \sigma^*\mathcal{L}(\Delta^i)$ = fiber of $\mathcal{L}|_{\sigma(\Delta^i)}$; $\forall p \in \Delta^i$, get rest^n map

$P_p^\sigma : \mathcal{L}_\sigma \xrightarrow{\sim} \mathcal{L}_{\sigma(p)} = (\sigma^*\mathcal{L})_p$

$p \in \tau \subset \sigma$: $P_\tau^\sigma = (P_p^\tau)^{-1} \circ P_p^\sigma : \mathcal{L}_\sigma \longrightarrow \mathcal{L}_\tau$

$$C_i(X, \mathcal{L}) = \left\{ \sum_\sigma l_\sigma \cdot \sigma \;\middle|\; \begin{array}{l} \Sigma \text{ is finite (modify for BM)} \\ l_\sigma \in \mathcal{L}_\sigma \end{array} \right\}$$

$$\partial_i\left(\sum_\sigma l_\sigma \sigma\right) = \sum_\sigma \sum_{\substack{\tau : \text{face} \\ \text{of } \sigma}} \pm \underbrace{P_\tau^\sigma(l_\sigma)}_{\text{only difference from normal homology.}} \cdot \tau$$

$$H_i(X, \mathcal{L}) = \operatorname{Ker}(\partial_i) / \operatorname{Im}(\partial_{i+1})$$

Ex: $H_i(\mathbb{C}^\times, \mathcal{L}_m) = ?$ [$\mathcal{L}_m$ = local system for monodromy $z \mapsto z^m$.]

e.g. $\mathbb{Z} \longrightarrow Aut(A) = A^\times$, $n \longmapsto \lambda^n$

$\mathbb{C}^\times \underset{\text{homotopy}}{\sim} S^1$, so (suffices to do S^1: $S^1 =$  has triangulation T w/ $\sigma : \Delta^1 \to S^1$.

$C_1^T(\mathbb{C}^\times, \mathcal{L}_m) = C_0^T(\mathbb{C}^\times, \mathcal{L}_m) = A$.

$$0 \longrightarrow A \xrightarrow{\lambda - Id_n} A \longrightarrow 0$$

the isomorphism matters!

$$H_0(\mathbb{C}^\times, \mathcal{L}_m) = \operatorname{Ker}(\zeta_m - 1) = \begin{cases} 0 & \text{if } m \neq 1 \\ k & \text{if } m = 1 \end{cases}$$

$$H_1(\mathbb{C}^\times, \mathcal{L}_m) = k/(\zeta_m - 1) = \begin{cases} 0 & \text{if } m \neq 1 \\ k & \text{if } m = 1 \end{cases}.$$

▷ Can define similarly for IH_i

▷ Have all similar formulas as H_i : MV seqs, [Poincaré duality], if fibers are v.s.: $IH_i(X, \mathcal{L}) \cong IH^i(X, \mathcal{L})^*$

▷ $C_*(X, \mathcal{L}) = C_*(X_H) \underset{G/H}{\otimes} M$ & $C^*(X, \mathcal{L}) = Hom^*_{G/H}(C_*(X_H), M)$.

$HC\operatorname{Ker}(\rho_2)$ ↗ ; M ↑ fiber

▷ $H^0(X; \mathcal{L}) \cong M^{\pi_1(X)}$.

Q: What does $IC_p^\bullet(\mathcal{L})$. look like as a complex of shvs?
e.g. What are its stalk? How does it interact w/ Whitney strata?

...

Prop: $x \in X_{n-k} - X_{n-k-1}$. Then

$$\mathcal{H}^i(IC_p^\bullet(\mathcal{L}))_x \cong \left\{ \begin{array}{ll} 0 & , \ i > p_k - n \\ IH^p_{-i-(n-k+1)}(L_x, \mathcal{L}) & , \ i \leq p_k - n \end{array} \right\} \cong I^{BM}H_i^p(U, \mathcal{L})$$
$$U \cong \mathbb{R}^{n-k} \times cL_x$$

Pf: Let $U \ni x$ be a nghd small enough s.t.

$$U \cong \mathbb{R}^{n-k} \times \operatorname{cone}(L_x), \quad \dim(L_x) = k-1$$

$$H^i(IC_p^\bullet(U, \mathcal{L})) \cong H_i(I^{BM}C^p_{-\bullet}(U, \mathcal{L})) \cong H_{i+n-k}(I^{BM}C^p_{-\bullet}(cL_x, \mathcal{L}))$$

$$\cong I^{BM}H^p_{-n+k-i}(cL_x, \mathcal{L}) \underset{\text{cone formula}}{=} \left\{ \begin{array}{ll} 0 & , -n+k-i < k-1-p_k+1 \\ & \Leftrightarrow p_k - n < i \\ IH^p_{-n+k-i-1} & , p_k - n \geq i \end{array} \right.$$

//

$$U_k = X - X_{n-k} \underset{\text{open}}{\subset} X$$

$$U_k \xhookrightarrow{j_k} U_{k+1} \xhookleftarrow{i_k} U_{k+1} - U_k = X_{n-k} - X_{n-k-1}$$

Prop: (a) $\mathcal{H}^{-i}(Rj_{k*}IC_p^\bullet|_{U_k})_x \cong I^{BM}H_i^p(U \cap U_k; A)$
for U disting. nghd (i.e. $U \cong \mathbb{R}^{n-k} \times cL_x$)

(b) $\mathcal{H}^j(IC_p^\bullet|_{U_{k+1}}) = 0$ if $j > p_k - n$ $\forall k \geq 2$.

(c) $IC_p^\bullet|_{U_{k+1}} \longrightarrow Rj_{k*}j_k^*(IC_p^\bullet|_{U_{k+1}})$ induces isoms.

$$\mathcal{H}^j(i_k^* IC_p^\bullet|_{U_{k+1}}) \xrightarrow{\sim} \mathcal{H}^j(i_k^* Rj_{k*}j_k^* IC_p^\bullet|_{U_{k+1}}) \quad \forall j \leq p_k - n.$$

IC equals its lower-* extension on $U_{k+1} - U_k$
up to degree $p_k - n$.

Lem: $IC_p^\bullet$ is constructible w.r.t. Whitney strat^n on X.

(d) [Pf:

Together these conditions form axioms:

⓪ - normalization ② - vanishing condition

① - lower bound ③ - attaching condition

(actually normal^{n} + lower bd are weaker...)

⓪ $\mathcal{A}^\bullet|_{U_2} \cong \mathcal{L}[n]$ (fixed $\mathcal{L}$: local system)

① $\mathcal{H}^j(\mathcal{A}^\bullet) = 0 \quad \forall j < -n$

② $\mathcal{H}^j(\mathcal{A}^\bullet|_{U_{k+1}}) = 0 \quad \forall j > p_k - n$

③ $\mathcal{H}^j(i_k^* \mathcal{A}^\bullet|_{U_{k+1}}) \cong \mathcal{H}^j(i_k^* R j_{k*} j_k^* \mathcal{A}^\bullet|_{U_{k+1}}) \quad \forall j \leq p_k - n$

These axioms will end up characterizing $IC_p^\bullet$!

→ <u>Deligne's Complex</u>: Define $\mathcal{G}_p^\bullet$ inductively by:

- On U_2, $\mathcal{G}_p^\bullet|_{U_2} = \underline{A}_{U_2}[n]$
- If we know $\mathcal{G}_p^\bullet|_{U_k}$, we put

$$\mathcal{G}_p^\bullet|_{U_{k+1}} = \underbrace{\tau_{\leq p_k - n}}_{\text{truncates cx into degs. } \leq p_k - n} R j_{k*} \mathcal{G}_p^\bullet|_{U_k}$$

<u>Thm</u>: If $\mathcal{A}^\bullet \in D^b(X)$ satisfies axioms ⓪–③, then $\mathcal{A}^\bullet \underset{\text{gls.}}{\cong} \mathcal{G}_p^\bullet$

<u>Pf</u>: Induct on U_k. True on U_2 trivially. Given $\mathcal{A}^\bullet|_{U_k} \cong \mathcal{G}_p^\bullet|_{U_k}$,

$$\mathcal{A}^\bullet|_{U_{k+1}} \overset{②}{\cong} \tau_{\leq p_k - n} \mathcal{A}^\bullet|_{U_{k+1}} \overset{\text{adjunction}}{\underset{⊛}{\longrightarrow}} \tau_{\leq p_k - n} R j_{k*} j_k^* (\mathcal{A}^\bullet|_{U_{k+1}})$$

$$\cong \tau_{\leq p_k - n} R j_{k*} (\mathcal{A}^\bullet|_{U_k}) \cong \tau_{\leq p_k - n} R j_{k*} (\mathcal{G}_p^\bullet|_{U_k}) =: \mathcal{G}_p^\bullet|_{U_{k+1}}.$$

[illegible] ⊛ is iso. bc we can apply ③ to truncation using ②

<u>Cor</u>. ⓪ + ③ $\Longrightarrow$ constructible.

$$IC_p^\bullet \cong \mathcal{G}_p^\bullet$$

<u>Rmk</u>:
- All works equally well w/ $\mathcal{L}$ in place of $\underline{A}_{U_2}$. (maybe just do this!)
- Note $A_{U_2}[n] = \mathbb{Q}_{U_2}$ [or $\mathcal{L} \otimes \mathbb{Q}_{U_2}$]

A different formulation of axiom (3):

$$i_*i^! \to id \to Rj_*j^* \xrightarrow{[1]} \xrightarrow{i^*} i^! \to i^* \to i^*Rj_*j^* \xrightarrow{[1]}$$

$i = i_k$, $j = j_k$: (3) $\iff \mathcal{H}^i(i_k^! A^\bullet | U_{k+1}) = 0 \quad \forall i \le p_k - n + 1$

For $x \in U_{k+1} \setminus U_k = X_{n-k} \setminus X_{n-k-1}$ & $f_x : \{x\} \hookrightarrow \underbrace{X_{n-k} - X_{n-k-1}}_{\text{sm mfd of dim } n-k}$

$f_x^!$ computed by $f_x^! \cong f_x^*(-)[-(n-k)]$.

(bc compute cpt coh. of a nghd of x in $X_{n-k} - X_{n-k-1}$.)

$$i_x^! A^\bullet \cong f_x^*(i_k^! A^\bullet)[-n+k]$$

So phrasing (3) in terms of stalks: $\overbrace{p_k - n + 1 + n - k}$ (shift from $f_x^!$)

(3') "Co-stalk vanishing": $\mathcal{H}^i(i_x^! A^\bullet) = 0 \quad \forall i < \overbrace{p_k - k + 1}$ & $\forall x \in X_{n-k} - X_{n-k-1}$

$\mathcal{H}^i(i_x^* A^\bullet) \cong \mathbb{H}^i(U_x, A^\bullet)$

$\mathcal{H}^i(i_x^! A^\bullet) \cong \mathbb{H}_c^i(U_x, A^\bullet)$

Consequences for IH hold for IC:

- ▷ Kunneth: $IC^\bullet_{X_1 \times X_2} \cong IC^\bullet_{X_1} \boxtimes IC_{X_2}$
- ▷ X: cx variety $\Rightarrow IC^\bullet_m \cong IC^\bullet_n$ (call this $IC^\bullet_X$)

Thm (P.D.): A: field. $p + q = t$ (complementary perversities).

Then $\exists$ qiso. $D_X(IC_q^\bullet)[n] \cong IC_p^\bullet$

Pf: need to show $D_X IC_q^\bullet[n]$ satisfies axioms for $IC_p^\bullet$.

(0) ($j: U_2 \hookrightarrow X$) $j^* D_X IC_q^\bullet[n] \cong D_{U_2} j^! IC_q^\bullet[n] \cong D_{U_2}(\underline{A}_{U_2}[n])[n] \cong D_{U_2}(\underline{A}_{U_2})$

$= \omega_{U_2} \cong \underline{A}_{U_2}[n]$.

(2) $x \in X_{n-k} - X_{n-k-1}$. $\mathcal{H}^j(i_x^! IC_q^\bullet) = 0 \quad \forall i \le q_k - k + 1$ by co-stalk vanishing.

so: $\mathcal{H}^j(D_X IC_q^\bullet[n])_x \cong \mathcal{H}^{j+n}(D_X IC_q^\bullet)_x \cong \mathbb{H}^{j+n}(U_x, D_X IC_q^\bullet)$

$\overset{UCT}{\cong} \mathbb{H}_c^{-j-n}(U_x, IC_q^\bullet)^* \cong \mathcal{H}^{-j-n}(i_x^! IC_q^\bullet)^* = 0$ for

$-j-n \le q_k - k + 1 \iff j \ge k - n - q_k - 1 = p_k - n + 1$

($q_k + p_k = k - 2$)

(3) sim. $\mathcal{H}^j(i_x^! D_X IC_q^\bullet[n]) \cong \mathcal{H}^{j+n}(i_x^! D_X IC_q^\bullet) = \mathbb{H}_c^{j+n}(U_x, D_X IC_q^\bullet) \overset{UCT}{\cong} \mathbb{H}^{j-n}(U_x, IC_q^\bullet)^*$

$\cong \mathcal{H}^{-j-i}(i_x^* IC_q^\bullet)^* = \mathcal{H}^{-j-i}(IC_q^\bullet)_x^* = 0$ for $-j-n > q_k - n \leftrightarrow j \le p_k - k + 1$

① Same as ②. //

Cor: ∃ nondeg. pairing

$$I^{BM}H_i^p(X,A) \otimes I H_{n-i}^q(X,A) \longrightarrow A .$$

Pf:
$$I^{BM}H_i^p(X,A) \cong \mathbb{H}^{-i}(X, IC_p^\bullet) \cong \mathbb{H}^{-i}(X, D_X IC_q^\bullet[n])$$
$$\cong \mathbb{H}^{n-i}(X, D_X IC_q^\bullet) \overset{UCT}{\cong} \mathbb{H}_c^{-n+i}(X, IC_q^\bullet)^* \cong I H_{n-i}^q(X,A)^* \;//$$

Small adaptation gives:

$$D_X(IC_q^\bullet(\mathcal{L}^\vee)[n]) \cong IC_p^\bullet(\mathcal{L})$$

& so also get:

$$I^{BM}H_i^p(X;\mathcal{L}^\vee) \cong I H_{n-i}^q(X,\mathcal{L})$$

Q: All of this still depends on a Whitney stratif^n.
Is $IC_p^\bullet$ independent of choice?

Thm: $\mathcal{X}_1, \mathcal{X}_2$: 2 Whitney strat^n's on X. Then

$$IC_{p,\mathcal{X}_1}^\bullet \underset{\text{qiso.}}{\simeq} IC_{p,\mathcal{X}_2}^\bullet .$$

Pf Sketch: ▷ Find equivaleat, intrinsic axioms that are equiv. to ours & show they characterizes $IC_p^\bullet$ indep. of strat^n.

Intrinsic Axioms: $\mathcal{A}^\bullet \in D(X)$:

$$\operatorname{supp}^j(\mathcal{A}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^* \mathcal{A}^\bullet) \neq 0\}}$$
$$\operatorname{cosupp}^j(\mathcal{A}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^! \mathcal{A}^\bullet) \neq 0\}}$$
$$p^{-1}(j) = \begin{cases} \min\{k : p(k) \geq j\} & , j \leq p(n) \\ \infty & , j > p(n) \end{cases}$$

⓪ (normal^n) $\exists U \subset X$ open s.t. $\mathcal{A}^\bullet|_U \cong \omega_U = \underline{A}_U[n]$ } same
① (lower bd) $\mathcal{H}^j(\mathcal{A}^\bullet) = 0 \quad \forall j < -n$
②′ (support) $\dim \operatorname{supp}^j(\mathcal{A}^\bullet) \leq n - p^{-1}(j+n) \quad \forall j > -n$
③ (cosupport) $\dim \operatorname{cosupp}^j(\mathcal{A}^\bullet) \leq n - q^{-1}(j) \quad \forall j < 0$. for q: compl. perversity

on back →