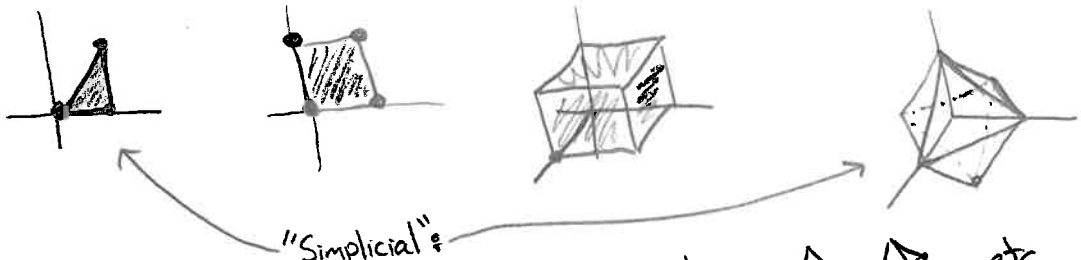
First Class:

-Def. A polytope, P, is the convex hull of a set of vertices in $\mathbb{R}^N$. ie. $v_1, \dots, v_k \in \mathbb{R}^N \Rightarrow \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; 0 \le \lambda_i \le 1 \right\}$

-Exs:



"Simplicial": all proper facets are simplexes $\triangle$, $\triangle$, etc.

Face vector: $f = (f_0, f_1, \dots, f_{d-1})$, f_i = # of i-dimensional faces

Q: Given a vector f, when is there an associated polytope? [Simpler: can we answer for what f there is an associated simplicial polytope?]

McMullen: Better to associate an h-vector to P.

→ Inductive definition (induction on faces of P)

→ For simplicial P, $h(P) = (h_0, \dots, h_d)$ are coeffs. of

$$h(P,t) = (t-1)^d + f_0(t-1)^{d-1} + \cdots + f_{d-1}$$

e.g. $P = \Delta_n$ a n-diml simplex $\Rightarrow h(\Delta_n, t) = t^d + t^{d-1} + \cdots + 1$

P = cube $\Rightarrow h(P,t) = t^3 + 5t^2 + 5t + 1$

Conjecture (McMullen) / Thm (Stanley): $h \in \mathbb{Z}_{\ge 0}^{d+1}$ is an h-vector for a d-diml simplicial polytope P iff

① $h_i = h_{d-i}$ $\forall i$

② $\exists$ graded $\mathbb{Q}$-alg. R with $R_0 = \mathbb{Q}$, R generated in deg 1, $\dim_{\mathbb{Q}} R_i = h_i - h_{i-1}$ $\forall 1 \le i \le \lfloor d/2 \rfloor$

Cor. P: simplicial $\Rightarrow$ $h(P)$ is unimodal

Stanley's idea: $P \longleftrightarrow$ "fans" $\longleftrightarrow$ toric variety $X_P / \mathbb{C}$

Thm: P: simplicial $\Rightarrow$ $h(P,t) = \sum_{k=0}^{d} \dim H^{2k}(X_P, \mathbb{Q}) \cdot t^k$ & $H^{odd}(X_P, \mathbb{Q}) = 0$

When P is not simplicial, X_P is not "$\mathbb{Q}$-smooth" & Stanley's theorem fails. But there is a fix!

② **Thm (Stanley):** For general rational P,

$$h(P,t) = \sum_{k=0}^{d} \dim \underbrace{IH^{2k}(X_P, \mathbb{Q})}_{\text{"Intersection homology"}} \cdot t^k$$

~ We will return later in the course to prove this. ~

What is "Intersection (co)homology"?

A: provides a good (co)homology theory for singular varieties

"good": we can do intersection theory with it.

Intersection Theory for smooth varieties:

X: smooth, oriented n-manifold, $H_n(X;\mathbb{Z}) \cong \mathbb{Z} \ni [X]$ fundamental class

Thm (Poincaré Duality). $- \cap [X] : H^i(X,\mathbb{Z}) \longrightarrow H_{n-i}(X,\mathbb{Z})$

So with rational coeffs:

$$\begin{array}{ccc} H^i(X,\mathbb{Q}) & \xrightarrow{\sim} & H_{n-i}(X,\mathbb{Q}) \\ \| & \text{universal coeffs. thm} & \| \\ H_i(X,\mathbb{Q})^* & & H^{n-i}(X,\mathbb{Q})^* \end{array}$$

Cor: $h_i = \dim_{\mathbb{Q}} H_i(X,\mathbb{Q}) = \dim_{\mathbb{Q}} H^{n-i}(X,\mathbb{Q})$ is a symmetric/palandromic sequence.

Better: can specify the isomorphism $H^{n-i}(X,\mathbb{Q}) \cong H^{n+i}(X,\mathbb{Q})$

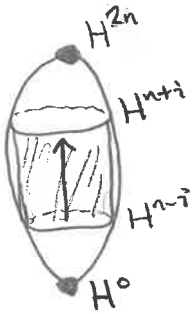
Assume $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ is a complex, projective variety.

Choose a generic hyperplane $H \subset \mathbb{P}^N$

$\rightsquigarrow [X \cap H] \in H_{2n-2}(X,\mathbb{Z}) \overset{\text{P.D.}}{\Longleftrightarrow} [H] \in H^2(X,\mathbb{Z})$

Thm (Hard Lefschetz). $H^{n-i}(X,\mathbb{Q}) \xrightarrow[\sim]{-\cup [H]^i} H^{n+i}(X,\mathbb{Q})$ is isom.

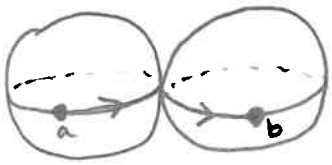
↳ Rmk on how "hard" Hard Lefschetz is: hard enough Lefschetz didn't prove it in his 1924 paper; Hodge did in 1941!



Cor. h_i are also unimodal.

③ This fails for singular spaces

Ex: $X = \mathbb{P}^1 \cup \mathbb{P}^1 \subseteq \mathbb{P}^2$



By Mayer-Vietoris:

$$H_i(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} & i=0 \\ 0 & i=1 \\ \mathbb{Q}\oplus\mathbb{Q} & i=2 \end{cases}$$ no Poincaré duality

$H^i(X,\mathbb{Q})^*$

What is going on? a & b want to be distinct 0-cycles, but they are connected by a 1-cocycle through the singularity.

IH will exclude this cocycle:

$$IH^i(X,\mathbb{Q})^* \cong IH_i(X,\mathbb{Q}) = \begin{cases} \mathbb{Q}\oplus\mathbb{Q} & i=0 \\ 0 & i=1 \\ \mathbb{Q}\oplus\mathbb{Q} & i=2 \end{cases}$$

But what if I only care about cohomology? IH has a sheafy analogue called perverse sheaves.

cohomology sheaf of a proper map $\in$ { perverse sheaves }

The category of perverse sheaves has atoms called "Intersection Cohomology" (IC) sheaves.

Thm [Decomposition Thm]. $f: X \to B$ a proper map between smooth varieties, then $Rf_* \overline{\mathbb{Q}}_\ell$ is a semisimple perverse sheaf, i.e. it is a direct sum of IC sheaves.

$\Longrightarrow$ If you know the supports of summands, you can go from general fibers to smooth fibers!

Underlies Springer theory, proof of Fundamental Lemma, Kazhdan-Luzstig theory, proof of P=W conjecture, ... many many more!