

1 Introduction

I take an algebro-geometric point of view towards understanding arithmetic phenomenon. A theme of my work is *duality* motivated by different facets of number theory and physics; for example:

- matching geometrizations of periods of automorphic forms and special values of L -functions on two sides of the Dolbeault relative Langlands duality [HLM25];
- a relative fundamental lemma in the case of $U_{2n}/GL_n(E)$ using relative Hitchin systems [HL26];
- comparing combinatorial data in the 3D mirror symmetry between affine Springer fibers and Hilbert schemes in the setting of torus links [AHM26].

1.1 Relative Langlands

The Langlands program predicts a correspondence between automorphic representations of a reductive group G and Galois representations into its dual group $\widehat{G}$. One way to study this correspondence is by matching their invariants. For example, on the automorphic side, we may consider *period integrals* which integrate over orbits of subgroups $H \subset G$. On the Galois side, period integrals match special values of L -functions attached to representations of $\widehat{G}$; see for example [Wal85, FJ93a, Ich08, GP92, II10, JR96]. A classical case is Hecke’s Mellin transform identity for a cusp form f [Hec36]:

$$\int_0^\infty f(iy)y^s \frac{dy}{y} = (2\pi)^{-s} \Gamma(s) L(s, f).$$

Recently, this philosophy has been unified by the burgeoning *relative Langlands* program developed by Ben-Zvi, Sakellaridis, and Venkatesh [BZSV24], which derives these identities from a *duality* between certain hyperspherical Hamiltonian G -varieties.

My work with Zhilin Luo and Benedict Morrissey studies a geometric form of this duality on moduli spaces of Higgs bundles [HLM25]. In this setting, periods and L -functions have counterparts as coherent sheaves, and their predicted equality becomes a Fourier–Mukai duality. In our work, we focus on the case where $M = T^*X$ for a homogeneous affine spherical variety $X = G/H$. In this case, the period sheaf is obtained from the conormal bundle to Bun_H in Bun_G . We denote this space by Higgs_X and call its objects *relative Higgs bundles*. Its natural morphism to Higgs_G is Lagrangian [GR18], giving an example of a *BAA brane* in mathematical physics. For symmetric spaces, these moduli spaces have been studied extensively in [Sch13, Sch15, Sch18b, GPPN23] and my work with Morrissey [HM25].

Hitchin systems offer two advantages for studying this duality. First, they make its mechanism explicit. The moduli of G -Higgs bundles admits a map to an affine space which generalizes the characteristic polynomial of a matrix. Its generic fibers are essentially abelian varieties, up to components and stabilizers, and duality is realized by the Fourier–Mukai transform between dual fibers. This allows us to formulate a precise comparison of period and L -sheaves (Conjecture 3) and prove it at the level of the symmetry groups of generic relative Hitchin fibers (see Theorem 4). The remaining translations between components lead to Hecke-theoretic questions that we pose in Project 5.

Second, the geometry of Higgs_X connects this duality to the invariant theory of G acting on T^*X . Information flows in both directions: Knop’s results on spherical varieties supply geometric input for relative duality, while the predicted duality reveals how the orbit structure of T^*X should determine the dual Hamiltonian $\widehat{M}$. For tempered X , whose generic stabilizers in T^*X are abelian, we propose an explicit construction of the dual symplectic representation from the branching of regular orbits. This gives an invariant-theoretic explanation for data originally defined through arithmetic examples. These formulas were also obtained independently by Ben-Zvi, Sakellaridis, and Venkatesh, who have since announced a proof in the strongly tempered case.

1.2 Relative Endoscopy

One of the striking aspects of the Langlands program is *functoriality*: Namely, relationships between dual groups $\widehat{G}_\epsilon \subset \widehat{G}$ predict natural relationships between automorphic representations of G_ϵ and G , even if the groups G_ϵ and G have no obvious relation with each other. Endoscopy, which arises roughly when $\widehat{G}_\epsilon$ is the centralizer of a semisimple element, is the first general case of such a principle. To prove it, Langlands proposed a trace formula, later developed by Arthur, which relates the traces of automorphic representations to orbital integrals [Lan83, Art81]. Comparing these orbital integrals across groups leads

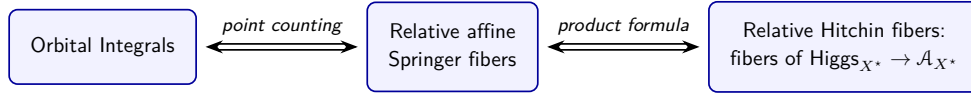


Figure 1: Relations between key concepts/moduli spaces in the proof of Theorem 9.

to the Langlands-Shelstad fundamental lemma, which was resolved in spectacular fashion by Ngô Bảo Châu using the geometry of the Hitchin fibration [Ngo06, Ngo10].

My work with Spencer Leslie develops this geometric approach in the relative setting. Building on Leslie’s formulation of endoscopy for symmetric spaces [Les24] and his work with Xiao and Zhang [LXZ25b, LXZ25a], we prove a fundamental lemma for the unitary symmetric space $X = \mathrm{U}_{2n}/\mathrm{GL}_{n,E}$, where E/F is an unramified quadratic extension of local fields [HL26]. This is the first geometric proof of a relative fundamental lemma for symmetric spaces. Our method also recovers Leslie’s fundamental lemma for $\mathrm{U}_{2n}/(\mathrm{U}_n \times \mathrm{U}_n)$, originally proved using p -adic harmonic analysis [Les25].

These identities supply local ingredients needed to stabilize Jacquet’s *relative trace formula*, which relates periods of automorphic forms to relative orbital integrals [JL85]. This is fundamental to the study of *distinction*: An automorphic representation is H -distinguished if its periods over H are not all zero. Comparisons of relative trace formulas can relate distinction to properties of L -functions. A model for such a relationship is Friedberg and Jacquet’s characterization of linear periods for $\mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$: for a cuspidal representation π with trivial central character, distinction is equivalent to π being of symplectic type and $L(\frac{1}{2}, \pi, \mathrm{std}) \neq 0$ [FJ93b].

Our proof interprets orbital integrals via point counts of moduli spaces that we call *relative affine Springer fibers (RASFs)*, which parametrize certain self-dual lattices. RASFs are finite dimensional projective ind-schemes which are finite modulo a lattice action; think for example of an infinite chain of projective lines. We study their cohomology by passing through a product formula to Higgs_X and using the global geometry of the Hitchin fibration. For example, it is already a deep result that builds on global techniques in [Chi22] to prove that RASFs are equidimensional and to compute their dimension.

The global argument in [HL26] draws from my work with Morrissey on the geometry of Higgs_X when X is a symmetric space [HM25]. In particular, a new phenomena is the divergence between the GIT quotient, which parametrizes closed G orbits in T^*X and our regular quotient, which parametrizes regular G orbits in T^*X . When X is a symmetric space, this discrepancy is completely determined by a *branching divisor*, which is the same object that appears in my work with Morrissey and Luo on duality for the relative Hitchin system [HLM25] (see §2.1).

2 Relative Langlands duality for Higgs bundles

2.1 The relative Langlands program built from invariant theory

Recall that for a well-behaved Hamiltonian G -variety M , [BZSV24] posits the existence of a dual Hamiltonian $\widehat{G}$ variety $\widehat{M}$ so that periods for M match L -functions for $\widehat{M}$ and vice versa. In this section, we focus on the question of how to define $\widehat{M}$ given a Hamiltonian G -variety $M = T^*X$ for a homogeneous, affine spherical variety X .

For $M = T^*X$, the dual Hamiltonian is determined by a dual group $\widehat{G}_X$, a symplectic $\widehat{G}_X$ representation S_X , and an Arthur $\mathfrak{sl}_2$. We will further restrict to the case when the Arthur $\mathfrak{sl}_2$ is trivial. Such X are called *tempered* as they conjecturally correspond to periods with tempered matrix coefficients. The group $\widehat{G}_X$ is well understood and combinatorially defined, see [Nad05, GN09, KS17, SV17]. However, the representation S_X remains mysterious; its construction in [BZSV24] is informed by arithmetic examples.

Question 1. How does one define S_X directly from $M = T^*X$ in a canonical way?

In [HLM25], we give a partial answer to Question 1. Consider the scheme $M//G$ parametrizing closed G orbits in M . In fact, by a theorem of Knop, $M//G \simeq \widehat{\mathfrak{g}}_X//\widehat{G}_X = \mathrm{Spec}k[\widehat{\mathfrak{g}}_X]^{\widehat{G}_X}$, which is isomorphic to an affine space [Kno90, Kno96]. A reduced divisor $\mathfrak{D} \subset \widehat{\mathfrak{g}}_X//\widehat{G}_X$ which is the image of weight hyperplanes for minuscule weights determines a unique symplectic representation $S_{\mathfrak{D}}$ of $\widehat{G}_X$ such that $2\mathfrak{D} = (\det d\rho_{\mathfrak{D}})$,

$$\begin{array}{ccc}
\mathcal{P}_M^{\text{dR}} = \mathcal{O}_{\text{Bun}_H} & D\text{-mod}(\text{Bun}_G) \xleftarrow{\sim} \text{IndCoh}(\text{Loc}_{\widehat{G}}) & \mathcal{L}_{\widehat{M}}^{\text{dR}} \\
& \downarrow \text{associated graded of a filtration} & \downarrow \text{"nonabelian Hodge"} \\
\mathcal{P}_M = \mathcal{O}_{\text{Higgs}_X} & \text{Coh}(\text{Higgs}_G) \xleftarrow{\sim} \text{Coh}(\text{Higgs}_{\widehat{G}}) & \mathcal{L}_{\widehat{M}} = \bigwedge^{\bullet} \text{DH}(\widehat{G}_X, S_X)
\end{array}$$

Figure 2: The 1 parameter degeneration of geometric Langlands. The top arrow is geometric Langlands; the bottom is the “Dolbeault” form that we study here. The blue

see [Ham26]. Note that symplectic representations arising in this way are constrained: For example, the determinant $\det d\rho_{\mathfrak{D}}$ always admits a square root.

Consider the map $\chi: M \rightarrow M//G$ taking an element m to the unique closed orbit in the closure $\overline{G \cdot m}$. We say that $m \in M$ is *regular* if the dimension of $G \cdot m$ is maximal. We define the *branching locus* $\mathfrak{D}_{\text{br}}$ by the closure

$$\mathfrak{D}_{\text{br}} = \overline{\{a \in M//G \mid \chi^{-1}(a) \text{ contains exactly 2 regular orbits}\}}.$$

In [HLM25], we propose the following conjecture/definition.

Definition 2. Suppose X is a tempered, affine spherical variety with no type N roots. If $\mathfrak{D}_{\text{br}}$ is a divisor in $M//G$, then $S_X = S_{\mathfrak{D}_{\text{br}}}$. Otherwise, S_X is trivial.

By stating the above as a definition, we are claiming that $\widehat{M}$ is well defined and agrees with the definition in [BZSV24]. With this definition, we prove concrete statements in [HLM25], and the elegance of this duality and its proofs can be taken as an argument that this definition is conceptually correct. We discuss these results in the section 2.2 next.

2.2 The semiclassical limit of relative Langlands

Fix a curve C over an algebraically closed field k of characteristic zero. Let $\text{Higgs}_G = T^*\text{Bun}_G$ denote the moduli of G -Higgs bundles, which for $G = \text{GL}_n$ classifies pairs (E, ϕ) where E is a rank n vector bundle and ϕ is a linear map $\phi: E \rightarrow E \otimes \Omega_C^1$. It was predicted in [HT03] and proved in [DP12] (over $\mathbb{C}$) and in [CZ17] (over general characteristic) that there is an equivalence of categories between coherent sheaves on Higgs_G and coherent sheaves on $\text{Higgs}_{\widehat{G}}$ generically.¹ This equivalence is given explicitly via a Fourier-Mukai transform on Hitchin fibers. This duality is known as the *semiclassical limit* of geometric Langlands, because it can be seen as a natural 1 parameter degeneration of the usual geometric Langlands correspondence (see Figure 2), and it can be seen as an example of SYZ mirror symmetry as envisaged in [SYZ01].

In [HLM25], we show that the relative Langlands program geometrizes to study specific sheaves on Higgs_G and $\text{Higgs}_{\widehat{G}}$ which are associated to Hamiltonian spaces M and $\widehat{M}$. Specifically, when $X = G/H$ is a homogeneous affine spherical variety with no type N roots, we associate to $M = T^*X$ the *period sheaf* $\mathcal{P}_M$ given by the structure sheaf of the moduli space of X -Higgs bundles Higgs_X , which we may view as the conormal bundle to $\text{Bun}_H \rightarrow \text{Bun}_G$. In the case where X is a symmetric space, these moduli spaces can be understood as fixed points of involutions on Higgs_G and were studied extensively in [Sch13, Sch15, Sch18a, GPPN23, HM25]. In this latter work with Morrissey [HM25], I study a “regular locus” in the moduli Higgs_X and show how its geometry is dictated by the same invariant theory that appears in Definition 2.

The semiclassical limit of relative Langlands predicts the Fourier-Mukai transform of $\mathcal{P}_M$ should be computed by an L -sheaf. In the case the dual representation S_X (see §2.1) admits a polarization $S_X = S_X^+ \oplus S_X^-$, the L -sheaf admits a description in terms of the *Dirac-Higgs bundle* $\text{DH}(\widehat{G}_X, S_X^+)$ studied in [FH24, Bla15, Hit16], which is a vector bundle on $\text{Higgs}_{\widehat{G}_X}$ obtained by the hypercohomology of the universal Higgs field induced by S_X^+ . In particular, $\mathcal{L}_{\widehat{M}} = \bigwedge \text{DH}(\widehat{G}_X, S_X^+)$.

¹“Generically” here means over a generic subset of the Hitchin base, where the Hitchin fibers are Beilinson 1-motives. It is a longstanding open problem to extend this equivalence further. See [Ari13, Li22, MRV19a, MRV19b] for some results in this direction.

Conjecture 3. Assume that X is a tempered, affine spherical variety with no type N roots. Suppose further that the dual symplectic representation S_X is polarized. Then, over a generic locus in the Hitchin base, the Fourier-Mukai transform of the period sheaf $\mathcal{P}_M$ is equal to the L -sheaf $\mathcal{L}_{\widehat{M}}$. More explicitly, the Fourier-Mukai transform of the structure sheaf of Higgs_X is the exterior algebra of the Dirac-Higgs bundle $\text{DH}(\widehat{G}_X, S_X^+)$.

Conjecture 3 can also be seen as a verification of predictions in mirror symmetry, which propose a correspondence between BAA branes (holomorphic Lagrangians) and BBB branes (hyperholomorphic sheaves), see for example [FPN23, FGPN24, BS14, BS16, FGPN21]. This correspondence was a central theme of the seminal work of Kapustin and Witten, which derived the categorical geometric Langlands conjectures from physical predictions in electro-magnetic duality [KW07].

We do not prove Conjecture 3 in full in [HLM25], but we prove the following partial result, which does not require the assumption that S_X be polarized.

Theorem 4. Define S_X as in Definition 2. Then, Conjecture 3 holds “up to translations.” That is, Conjecture 3 holds if we replace Higgs_X with a group of symmetries that acts simply transitively on it, and if we replace $\text{DH}(\widehat{G}_X, S_X^+)$ with a pushforward of a trivial vector bundle.

To illustrate the distance between Conjecture 3 and Theorem 4, consider the representative case of $X = \text{GL}_{2n} / \text{GL}_n \times \text{GL}_n$. For a generic point, a , in the Hitchin base, there is a smooth curve C_a embedded in the total space of Ω_C^1 such that the fiber of the Hitchin fibration is $\text{Pic}(C_a)$ and the morphism $\text{Higgs}_X \rightarrow \text{Higgs}_G$ is the map

$$\coprod_{D \subset C_a \cap \{\lambda=0\}} \text{Pic}(C_a/\iota) \rightarrow \text{Pic}(C_a), \quad (\mathcal{L})_D \mapsto q^* \mathcal{L} \otimes \mathcal{O}(D),$$

with $\{\lambda = 0\}$ denoting the zero section. Theorem 4 computes the Fourier-Mukai instead of pushforward of the structure sheaf along $(\mathcal{L})_D \mapsto q^* \mathcal{L}$ above.

2.3 Ongoing and future work

With Zeyu Wang, we plan to carry out the following:

Project 5. Complete the proof of Conjecture 3 using Theorem 4.

Our strategy is to understand the discrepancy between Conjecture 3 and Theorem 4 in terms of *Hecke operators*. Namely, we have a correspondence stack Hecke_X which classifies tuples (x_1, x_2, ξ) where x_1, x_2 are in Higgs_X and $\xi: x_1 \dashrightarrow x_2$ is an elementary modification of Higgs bundles. There are natural projections

$$\text{Higgs}_X \xleftarrow{\pi_1} \text{Hecke}_X \xrightarrow{\pi_2} \text{Higgs}_X.$$

We have the M -Hecke operator on coherent sheaves on Higgs_X given by the correspondence $\mathcal{F} \mapsto \pi_{2,*} \pi_1^*(\mathcal{F})$. This has the effect of pulling back along translations between components of Higgs_X . We expect M -Hecke operators to be compatible with $\widehat{M}$ -Wilson operators, which are tensor operations by certain vector bundles on $\text{Higgs}_{\widehat{G}}$ coming from a Clifford algebra: Namely, the Dirac-Higgs bundle $\text{DH}(\widehat{G}_X, S_X)$ admits a quadratic form coming from the symplectic structure on S_X , and we can therefore form the Clifford algebra $\text{Cl}(\text{DH}(\widehat{G}_X, S_X))$. We have an identification

$$\text{End} \left(\bigwedge \text{DH}(\widehat{G}_X, S_X^+) \right) = \text{Cl}(\text{DH}(\widehat{G}_X, S_X)),$$

so that $\text{Cl}(\text{DH}(\widehat{G}_X, S_X))$ acts naturally on $\mathcal{L}_{\widehat{M}}$ by tensor operations. A similar Clifford algebra plays an important role in computing higher order derivatives of L -functions; see for example [YZ17, YZ19, LW26]. We propose for Project 5:

- If S_X is polarized, then we expect the existence of a section of the Hitchin map for Higgs_X . Theorem 4 implies duality over this section.
- Show that the modules $\mathcal{P}_M$ of $\text{End}_*(\mathcal{P}_M)$ and $\mathcal{L}_{\widehat{M}}$ of $\text{End}_{\otimes}(\mathcal{L}_{\widehat{M}})$ are generated in degree 1. For example, this is clear for $\mathcal{L}_{\widehat{M}}$.
- Show that M -Hecke- $\widehat{M}$ -Wilson compatibility implies that the equivalence over the section is compatible with the L^2 action, and then generate the full equivalence from this.

The author notes that (surprisingly) the compatibility even between usual Hecke and Wilson operators has not appeared in the literature in general characteristic, and it is a part of this project to write this compatibility down for miniscule weights.

We further expect to extend Conjecture 3 to an L^2 -duality:

Project 6. Prove that there exists a Fourier-Mukai equivalence between the sheaf of translations, or M -Hecke sheaf, $\text{End}_*(\mathcal{P}_M)$ and the Clifford algebra $\text{Cl}(\text{DH}(\widehat{G}_X, S_X))$ which induces an equivalence of algebra objects. Moreover, this equivalence should be compatible with the module actions on both sides of Conjecture 3 in the polarized case.

Project 6 follows in the polarized case from Conjecture 3, but we envision a uniform proof independent of polarization. In particular, we expect that Project 6 is intimately connected to the geometry of the cotangent bundle of the RTF stack considered in [Sak23].

In another direction, over positive characteristic the relative duality statement of Conjecture 3 has potential applications to the de Rham setting of [BZSV24]. We propose:

Project 7. In positive characteristic, generalize the arguments of [CZ17] to deduce a “generic” form of the global conjectures in [BZSV24].

The main idea of this direction is that the nonabelian Hodge correspondence, which connects between the Dolbeault and de Rham moduli spaces, can be made into an algebraic equivalence using the theory of Azumaya algebras. In particular, statements of duality between Higgs_G and $\text{Higgs}_{\widehat{G}}$ can be transported to categorical statements of duality in de Rham geometric Langlands. This is particularly interesting as it is a *categorical* equivalence, not just a matching of sheaves.

Finally, we record a more speculative research direction. Consider the perverse cohomology sheaf ${}^p\mathcal{H}^k(\text{R}f_{G,*}\overline{\mathbb{Q}}_\ell)$ as a D module on the Hitchin base, which carries a natural good filtration. A recent result of Maulik, Shen, and Yin proves that the associated graded of this filtration is the Fourier-Mukai dual of the sheaf $\Omega_{\text{Higgs}_{\widehat{G}}}^k$ [MSY26].

Project 8. When $f: \text{Higgs}_X \rightarrow \mathcal{A}_X$ is the relative Hitchin fibration associated to a spherical variety X , interpret the associated graded of ${}^p\mathcal{H}^k(\text{R}f_*\overline{\mathbb{Q}}_\ell)$ in terms of the dual Hamiltonian.

We suspect that the answer is related to the Fourier-Mukai dual of $\bigwedge^k \text{DH}(\widehat{G}_X, S_X^\perp)$. Resolving Project 8 may carry further implications for the theory of relative endoscopy, which we discuss in the next section. Namely, the supports of simple constituents in the perverse decomposition of $\text{R}f_*\overline{\mathbb{Q}}_\ell$ controls the endoscopy for X . It would be conceptually satisfying to tie this decomposition to behavior of the dual Hamiltonian, and doing so may shed light on how to generalize the conjectures of [Les24] beyond the case of symmetric spaces.

3 Fundamental lemmas via relative Hitchin systems

3.1 The fundamental lemma

Let F be a local field; for example you may think about $F = \mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$. Fix an element $\gamma \in \mathfrak{g}(F)$. The phenomena of *endoscopy* occurs when there exists $\gamma' \in \mathfrak{g}(F)$ which is conjugate to γ over $G(\overline{F})$ but not $G(F)$. For example, for $G = \text{SL}_2$, $\mathfrak{g}(\mathbb{R}) \simeq \mathbb{R}^3$, the $G(\mathbb{C})$ orbits are hyperbolas $z^2 - xy = d$ for $d \in \mathbb{R}$, which consist of two distinct $G(\mathbb{R})$ orbits.

In my work with Spencer Leslie [HL26], we consider a variant of this problem: Let $X = G/H$ be a symmetric space over a local field F , with H the fixed points of an involution θ on G . The involution θ induces a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ is the (-1) eigenspace of θ . For $\gamma \in \mathfrak{h}^\perp(F)$ and a compactly supported function f , we consider the *orbital integral*

$$\text{Orb}_{X,\gamma}(f) := \int_{H_\gamma \backslash H(F)} f(h^{-1}\gamma h) \frac{dh}{dt}.$$

As $\gamma' \in \mathfrak{h}^\perp(F)$ ranges over $H(\overline{F})$ conjugacy classes of γ , the orbital integrals $\text{Orb}_{X,\gamma'}$ form a basis for the $H(F)$ invariant distributions on the geometric conjugacy class of γ . There is a natural Fourier basis $\{\text{Orb}_{X,a}^\kappa\}$ whose terms $\text{Orb}_{X,a}^\kappa$ are weighted sums of orbital integrals $\text{Orb}_{X,\gamma'}$ and which only depend on

the conjugacy class a of γ . In particular, there is a unique term called the *stable orbital integral* $\text{SOrb}_{X,\gamma}$, which sums over all $\text{Orb}_{X,\gamma'}$ without weights.

For every κ indexing the basis $\text{Orb}_{X,a}^\kappa$, [Les24] produces an endoscopic symmetric variety X_κ over F . He posits that there exists explicit functions $\Delta_X(a)$ and transfers of functions such that

$$\text{Orb}_{X,a}^\kappa(f) = \Delta(a) \cdot \text{SOrb}_{X^\kappa,a}(f_\kappa). \quad (3.1)$$

Let E/F be an unramified quadratic extension of local fields, and let U_{2n} denote the unitary group of rank $2n$ for the extension. We prove (3.1) in the following case.

Theorem 9. *Consider the symmetric space $X = \text{U}_{2n}/\text{GL}_{n,E}$. Then, for any conjugacy class $a \in (\mathfrak{h}^\perp // H)(F)$ of a regular, semisimple element in $\mathfrak{h}^\perp$, (3.1) holds.*

Our methods also recover the case of $\text{U}_{2n}/\text{U}_n \times \text{U}_n$, which was proved via p -adic methods in [Les25].

We can be more explicit in this case: Let $X_r^S = \text{U}_{2r}/\text{GL}_{r,E}$ and let $X_r^U = \text{U}_{2r}/\text{U}_r \times \text{U}_r$. Then, endoscopic spaces for $X = X_n^S$ are of the form

$$X^\kappa = X_r^S \times X_s^U = (\text{U}_{2r}/\text{GL}_{r,E}) \times (\text{U}_{2s}/\text{U}_s \times \text{U}_s)$$

for $r + s = n$. In particular, one of the endoscopic varieties is X_n^U , which recovers the stable orbital integrals for the unitary Friedberg-Jacquet variety.

3.2 Moduli spaces and the proof of Theorem 9

The proof of Theorem 9 goes through the geometry of Higgs_X . Namely, by an argument in motivic integration, one can reduce to proving Theorem 9 under the assumption that $F = \mathbb{F}_q((\varpi))$ is the field of Laurent series with finite coefficient field, and it suffices to prove (3.1) after evaluating both sides at the indicator function of the power series ring $\mathbf{1}_{\mathfrak{h}^\perp(\mathcal{O}_F)}$. Under these simplifications, the transfer factor Δ becomes an explicit power of q .

The simplified integrals on each side of (3.1) are equal to point counts on *relative affine Springer fibers* for X_n^S and X^κ , respectively, which we define now. In particular, to an n -tuple $a = (a_2, \dots, a_{2n})$ in $\mathcal{O}_F^n$, we may form the E algebra

$$R_a(E) = E[t]/(t^{2n} + a_2 t^{2n-2} + \dots + a_{2n-2} t^2 + a_{2n})$$

with the obvious sub- $\mathcal{O}_E$ -algebra $R_a(\mathcal{O}_E)$. Then $R_a(E)$ carries natural involutions $\iota(t) = -t$ and σ from the Galois action of E/F . We say $\Lambda \subset R_a(E)$ is an $R_a(\mathcal{O}_E)$ -lattice if it is stable under the $R_a(\mathcal{O}_E)$ action and it is free of rank n as an $\mathcal{O}_E$ -module. There are natural Hermitian forms h^S and h^U on $R_a(E)$ constructed canonically via trace pairings. For $\star \in \{S, U\}$, we define

$$\mathcal{M}^\star(a) := \{R_a(\mathcal{O}_E) \text{ lattices } \Lambda \text{ such that } \iota(\Lambda) = \Lambda \text{ and } \Lambda \text{ is self dual with respect to } h^\star\}$$

It turns out that the spaces $\mathcal{M}^\star(a)$ are representable by locally finite type, projective schemes, and that, via an intricate argument modeled on the work of Jingren Chi [Chi22], $\mathcal{M}^\star(a)$ is equidimensional. We call these spaces *relative affine Springer fibers (RASFs)*.

To compute the cohomology of RASFs, we globalize these moduli spaces to a unitary version of Higgs_X . Namely, fix a curve C with connected, étale double cover $C' \rightarrow C$, and we consider the moduli of unitary Higgs bundles $\text{Higgs}_{\text{U}_n}$ of Laumon and Ngô [LN08], which classifies triples $(\mathcal{E}, \phi, h)$ where $(\mathcal{E}, \phi)$ is a GL_n Higgs bundle on C' and $h: \mathcal{E} \rightarrow \sigma^* \mathcal{E}^\vee$ is a Hermitian structure on $(\mathcal{E}, \phi)$. There are natural involutions θ^S and θ^U on $\text{Higgs}_{\text{U}_n}$ given by

$$\theta^\star(\mathcal{E}, \phi, h) = \begin{cases} (\mathcal{E}, -\phi, -h) & \text{if } \star = S \\ (\mathcal{E}, -\phi, h) & \text{if } \star = U \end{cases}.$$

Then $\text{Higgs}_{X^\star}$ for $\star \in \{S, U\}$ is the fixed point stack $\theta^\star$. The global and local worlds communicate through a *product formula*, which says that $\text{Higgs}_{X^\star}$ can be glued from quotients of RASFs $\mathcal{M}^\star(a)$. In particular, one can pass information back and forth between the local and global moduli spaces (see Figure 3.2), and Theorem 9 becomes a question of computing the cohomology of $\text{Higgs}_{X^\star}$.

For both the Siegel and unitary cases, there is a Hitchin map $f^\star: \text{Higgs}_{X^\star} \rightarrow \mathcal{A}$ where $\mathcal{A}_{X^\star}$ is an affine space. After an étale base change which we exclude here,² there is a natural decomposition of the cohomology $Rf_*^S \overline{\mathbb{Q}}_\ell$ into κ constituent pieces, $Rf_*^S \overline{\mathbb{Q}}_\ell = \bigoplus_\kappa (Rf_*^S \overline{\mathbb{Q}}_\ell)_\kappa$. We call the stable constituent when $\kappa = 1$ by $(Rf_*^S \overline{\mathbb{Q}}_\ell)_{\text{st}}$. The geometric manifestation of Theorem 9 is the following.

²We must restrict to a suitable elliptic locus and base change to a cameral cover.

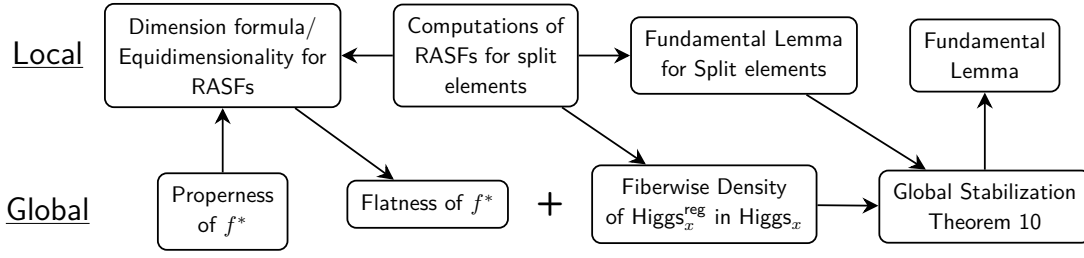


Figure 3: A summary of the how information is passed between local and global moduli spaces in the proof of Theorem 9.

Theorem 10. *There are canonical closed embeddings on Hitchin bases $\nu_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{A}$, and for any κ , we have an isomorphism of perverse sheaves over $\bar{k}$*

$$\nu_\epsilon^*(Rf_*^S \overline{\mathbb{Q}}_\ell)_\kappa[2r_\epsilon(D)](r_\epsilon(D)) = (f_*^\kappa \overline{\mathbb{Q}}_\ell)_{\text{st}} \quad (3.2)$$

which induces an isomorphism of semisimplifications over k .

The proof of Theorem 10 is a delicate argument using the Ngô’s support theorem [Ngo10] combined with a study of the geometry of regular Higgs bundles. Both sides of (3.2) are direct sums of shifts of semisimple perverse sheaves by the BBDG decomposition theorem [BBDG18]. We use deformation theory and the geometry of the branching divisor considered in §2.1 to show there are just enough regular Higgs bundles in just enough Hitchin fibers to constrain the supports of the simple constituents. This allows us to use a global induction argument to conclude Theorem 9 from some very small examples.

3.3 Ongoing and future work

The conjectures of Leslie include far more symmetric varieties than the two examples for which a fundamental lemma has now been proven. However, thus far, a balance has to be struck for many of the global arguments we make to work. In this section, we discuss some of the difficulties as well as current and future projects in this direction.

The proof of Theorem 10 uses techniques that are adapted specifically to the case of *abelian* fibrations; therefore, one of the most fundamental difficulties in adapting the geometric machinery is the non-temperedness of X . We illustrate this with the example $X = \text{GL}_{2n}/\text{Sp}_{2n}$. Work of Hitchin and Schaposnik prove that there is a family of curves $\mathcal{C} \rightarrow \mathcal{A}_X$ such that the fiber of the Hitchin map at $a \in \mathcal{A}_X$ is a compactification of the space of rank 2 vector bundles on $\mathcal{C}_a$ [HS14]. We propose to study this example, which has a particularly simple classification of supports.

Project 11. Show that in the case $X = \text{GL}_{2n}/\text{Sp}_{2n}$, all simple constituents of $f_{X,*} \overline{\mathbb{Q}}_{\ell,X}$ have full support.

There is a naive way to bound the supports using the action of the center of the centralizers, which is abelian. However, this does not constrain the supports enough to rule out smaller supports. Instead, we hope to apply more general support theorems, such as that of Migliorini and Shende [MS18].

In another direction, we wish to study periods of more general spherical varieties. In this case, the most relevant obstruction is the failure of flatness of the Hitchin morphism. In particular, this failure reflects the fact that we do not yet know a dimension formula for such relative affine Springer fibers. In [Yun16], Yun considers such a case by using a unitary version of $X = \text{GL}_n \times \text{GL}_{n+1}/\text{GL}_n$ by replacing Ngô’s support theorem with semismallness of the Hitchin map. This semismallness is a reflection of the fact that the generic stabilizers of $H = \text{GL}_n$ acting on $\mathfrak{h}^\perp$ are trivial, which is referred to as the condition that X be *strongly tempered*.

Project 12. Consider orbital integrals of more general spherical varieties. More explicitly:

1. For strongly tempered, smooth, affine spherical variety X with nontrivial dual symplectic representation S_X , adapt Yun’s argument to deduce fundamental lemmas.
2. For general tempered X , unify semi-smallness with Ngô’s support theorem.

Some evidence for the second proposed direction above comes from the factorization of the Hitchin map over the regular locus: In [HM25], we show that there is a large open set $\text{Higgs}_X^{\text{reg}} \subset \text{Higgs}_X$ which factors $\text{Higgs}_X^{\text{reg}} \rightarrow \mathcal{A}_X^{\text{reg}} \rightarrow \mathcal{A}_X$, where the latter map is an étale cover. If one could extend this factorization so the latter map is semismall and the former map captures the full cohomology of the fibers, Project 12 would be realized.

In another direction, we may hope to approach Project 12 using p -adic integration, which replaces the structure of perverse sheaves and the support theorem with more elementary integration along fibers. Such methods were used successfully to re-prove the Fundamental Lemma for Lie algebras and to extract deeper invariants from the Hitchin system to verify conjectures in mirror symmetry [GWZ20].

We also believe that our present arguments should be able to deduce cases that fall outside of the traditional boundaries of the relative Langlands program. We wish to consider one particular natural case of interest.

Project 13. Prove an endoscopic Fundamental Lemma for the case $X = \text{SL}_n / \text{SO}_n$. In particular, relate the corresponding endoscopic terms to the generalized dual Hamiltonian.

The case $X = \text{SL}_n / \text{SO}_n$ is a spherical variety with type N roots, meaning that its centralizers are disconnected. On the dual side, this reflects the fact that periods should be related to L -functions attached to metaplectic groups—see for example [Jac91, CO12]—and that the dual Hamiltonian carries stacky structure. However, its relative Hitchin fibration is perfectly well behaved, and so is well suited to prove formulas involving orbital integrals. I am actively working on this case.

Finally, I will end this section with an easier example, which I expect to follow from similar methods to those used in [HL26]. This project would be a wonderful opportunity for collaboration with a motivated graduate student interested in learning how to apply Hitchin systems to prove fundamental lemmas.

Project 14. Prove an endoscopic Fundamental Lemma for the case $X = \text{U}_{2n+1} / \text{U}_n \times \text{U}_{n+1}$.

While $\text{U}_{2n+1} / \text{U}_n \times \text{U}_{n+1}$ is tempered and looks nominally similar to the [Les25], it behaves geometrically more like a bridge case between tempered and non-tempered symmetric spaces. However, I expect only moderate difficulty in proving a fundamental lemma in this case.

References

- [AHM26] Pablo Boixeda Alvarez, Thomas Hameister, and Xinchun Ma, *Affine Springer fibers and Hilbert schemes: torus link case*, 2026.
- [Ari13] Dima Arinkin, *Autoduality of compactified Jacobians for curves with plane singularities*, J. Algebraic Geom. **22** (2013), no. 2, 363–388.
- [Art81] James Arthur, *The trace formula in invariant form*, Annals of Mathematics **114** (1981), no. 1, 1–74.
- [BBDG18] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber, *Faisceaux pervers*, Astérisque (2018), no. 100, vi+180.
- [Bla15] Jakob Lindblad Blaavand, *The Dirac-Higgs bundle*, Ph.D. thesis, University of Oxford, 2015, Available from <http://people.maths.ox.ac.uk/hitchin/files/StudentsTheses/blaaavand.pdf>.
- [BS14] David Baraglia and Laura P. Schaposnik, *Higgs bundles and (A, B, A) -branes*, Comm. Math. Phys. **331** (2014), no. 3, 1271–1300.
- [BS16] ———, *Real structures on moduli spaces of Higgs bundles*, Adv. Theor. Math. Phys. **20** (2016), no. 3, 525–551.
- [BZSV24] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh, *Relative Langlands duality*, 2024, arXiv preprint, arXiv:2409.04677.
- [Chi22] Jingren Chi, *Geometry of Kottwitz-Viehmann varieties*, J. Inst. Math. Jussieu **21** (2022), no. 1, 1–65.
- [CO12] Gautam Chinta and Omer Offen, *Orthogonal period of a $GL_3(\mathbb{Z})$ Eisenstein series*, Representation theory, complex analysis, and integral geometry, Birkhäuser/Springer, New York, 2012, pp. 41–59.
- [CZ17] Tsao-Hsien Chen and Xinwen Zhu, *Geometric Langlands in prime characteristic*, Compos. Math. **153** (2017), no. 2, 395–452.
- [DP12] Ron Donagi and Tony Pantev, *Langlands duality for Hitchin systems*, Inventiones mathematicae **189** (2012), no. 3, 653–735.
- [FGOPN21] Emilio Franco, Peter B. Gothen, André Oliveira, and Ana Peón-Nieto, *Unramified covers and branes on the Hitchin system*, Adv. Math. **377** (2021), Paper No. 107493, 61.

- [FGOPN24] ———, *Narasimhan-Ramanan branes and wobbly Higgs bundles*, *Internat. J. Math.* **35** (2024), no. 9, Paper No. 2441006, 27.
- [FH24] Emilio Franco and Robert Hanson, *The Dirac-Higgs complex and categorification of (BBB)-branes*, *Int. Math. Res. Not. IMRN* (2024), no. 19, 12919–12953.
- [FJ93a] Solomon Friedberg and Hervé Jacquet, *Linear periods*, *Journal für die reine und angewandte Mathematik* **443** (1993), 91–140.
- [FJ93b] Solomon Friedberg and Hervé Jacquet, *Linear periods*, *J. Reine Angew. Math.* **443** (1993), 91–139.
- [FPN23] Emilio Franco and Ana Peón-Nieto, *Branes on the singular locus of the Hitchin system via Borel and other parabolic subgroups*, *Math. Nachr.* **296** (2023), no. 5, 1803–1841.
- [GN09] Dennis Gaitsgory and David Nadler, *Hecke operators on quasimaps into horospherical varieties*, *Doc. Math.* **14** (2009), 19–46.
- [GP92] Benedict H. Gross and Dipendra Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* , *Canadian Journal of Mathematics* **44** (1992), no. 5, 974–1002.
- [GPPN23] Oscar García-Prada and Ana Peón-Nieto, *Abelianization of Higgs bundles for quasi-split real groups*, *Transform. Groups* **28** (2023), no. 1, 285–325.
- [GR18] Victor Ginzburg and Nick Rozenblyum, *Gaiotto’s Lagrangian subvarieties via derived symplectic geometry*, *Algebr. Represent. Theory* **21** (2018), no. 5, 1003–1015.
- [GWZ20] Michael Groechenig, Dimitri Wyss, and Paul Ziegler, *Geometric stabilisation via p -adic integration*, *Journal of the American Mathematical Society* **33** (2020), no. 3, 807–873.
- [Ham26] Thomas Hameister, *Note on symplectic representations and divisors*, 2026, Accessed at <https://hameist.github.io/repssanddivisors.pdf>.
- [Hec36] E. Hecke, *über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, *Math. Ann.* **112** (1936), no. 1, 664–699.
- [Hit16] Nigel Hitchin, *Higgs bundles and characteristic classes*, *Arbeitstagung Bonn 2013*, *Progr. Math.*, vol. 319, Birkhäuser/Springer, Cham, 2016, pp. 247–264.
- [HL26] Thomas Hameister and Spencer Leslie, *Relative Hitchin fibrations and endoscopy for unitary Friedberg–Jacquet varieties*.
- [HLM25] Thomas Hameister, Zhilin Luo, and Benedict Morrissey, *Relative Dolbeault geometric Langlands via the regular quotient*, 2025, To appear in *Selecta Mathematica*. Accessed at <https://arxiv.org/abs/2409.15691>.
- [HM25] Thomas Hameister and Benedict Morrissey, *The Hitchin fibration for symmetric pairs*, *Adv. Math.* **482** (2025), Paper No. 110560, 68.
- [HS14] Nigel Hitchin and Laura P. Schaposnik, *Nonabelianization of Higgs bundles*, *J. Differential Geom.* **97** (2014), no. 1, 79–89.
- [HT03] Tamás Hausel and Michael Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*, *Invent. Math.* **153** (2003), no. 1, 197–229.
- [Ich08] Atsushi Ichino, *Trilinear forms and the central values of triple product L -functions*, *Duke Mathematical Journal* **145** (2008), no. 2, 281–307.
- [II10] Atsushi Ichino and Tamotsu Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture*, *Geometric and Functional Analysis* **19** (2010), no. 5, 1378–1425.
- [Jac91] Hervé Jacquet, *Représentations distinguées pour le groupe orthogonal*, *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), no. 13, 957–961.
- [JL85] H. Jacquet and K. F. Lai, *A relative trace formula*, *Compositio Math.* **54** (1985), no. 2, 243–310.
- [JR96] Hervé Jacquet and Stephen Rallis, *Uniqueness of linear periods*, *Compositio Mathematica* **102** (1996), no. 1, 65–123.
- [Kno90] Friedrich Knop, *Weylgruppe und Momentabbildung*, *Invent. Math.* **99** (1990), no. 1, 1–23.
- [Kno96] ———, *Automorphisms, root systems, and compactifications of homogeneous varieties*, *J. Amer. Math. Soc.* **9** (1996), no. 1, 153–174.
- [KS17] F. Knop and B. Schalke, *The dual group of a spherical variety*, *Trans. Moscow Math. Soc.* **78** (2017), 187–216.
- [KW07] Anton Kapustin and Edward Witten, *Electric-magnetic duality and the geometric Langlands program*, *Commun. Number Theory Phys.* **1** (2007), no. 1, 1–236.
- [Lan83] Robert P. Langlands, *Les débuts d’une formule des traces stable*, *Publications Mathématiques de l’Université Paris VII*, vol. 13, Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983.
- [Les24] Spencer Leslie, *Symmetric varieties for endoscopic groups*, 2024.
- [Les25] Spencer Leslie, *The endoscopic fundamental lemma for unitary Friedberg–Jacquet periods*, *Ann. of Math.* (2) **201** (2025), no. 3, 551–645.

- [Li22] Mao Li, *The Poincaré line bundle and autoduality of Hitchin fibers*, Selecta Math. (N.S.) **28** (2022), no. 3, Paper No. 49, 52.
- [LN08] Gérard Laumon and Bao Châu Ngô, *Le lemme fondamental pour les groupes unitaires*, Ann. of Math. (2) **168** (2008), no. 2, 477–573.
- [LW26] Shurui Liu and Zeyu Wang, *Higher period integrals and derivatives of L-functions*, 2026.
- [LXZ25a] Spencer Leslie, Jingwei Xiao, and Wei Zhang, *Unitary Friedberg-Jacquet periods and their twists: Fundamental lemmas*, 2025.
- [LXZ25b] ———, *Unitary Friedberg-Jacquet periods and their twists: Relative trace formulas*, 2025.
- [MRV19a] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fourier-Mukai and autoduality for compactified Jacobians. I*, J. Reine Angew. Math. **755** (2019), 1–65.
- [MRV19b] ———, *Fourier-Mukai and autoduality for compactified Jacobians, II*, Geom. Topol. **23** (2019), no. 5, 2335–2395.
- [MS18] Luca Migliorini and Vivek Shende, *Higher discriminants and the topology of algebraic maps*, Algebr. Geom. **5** (2018), no. 1, 114–130.
- [MSY26] Daveshe Maulik, Junliang Shen, and Qizheng Yin, *Fourier-Mukai transforms and the decomposition theorem for integrable systems*, Algebr. Geom. **13** (2026), no. 2, 266–304.
- [Nad05] David Nadler, *Perverse sheaves on real loop Grassmannians*, Invent. Math. **159** (2005), no. 1, 1–73.
- [Ngo06] Bao Chau Ngo, *Fibration de Hitchin et endoscopie*, Invent. Math. **164** (2006), no. 2, 399–453.
- [Ngo10] ———, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169.
- [Sak23] Yiannis Sakellaridis, *Spherical varieties, functoriality, and quantization*, Proceedings of the International Congress of Mathematicians 2022, vol. 4, EMS Press, 2023, pp. 2998–3037.
- [Sch13] Laura P. Schaposnik, *Spectral data for G-Higgs bundles*, Ph.D. thesis, 2013, Thesis (D.Phil.)–University of Oxford (United Kingdom), p. 1.
- [Sch15] ———, *Spectral data for $U(m, m)$ -Higgs bundles*, Int. Math. Res. Not. I
- [Sch18a] L Schaposnik, *An introduction to spectral data for Higgs bundles*, The geometry, topology and physics of moduli spaces of Higgs bundles (2018), 65–101.
- [Sch18b] Laura P. Schaposnik, *A geometric approach to orthogonal Higgs bundles*, Eur. J. Math. **4** (2018), no. 4, 1390–1411.
- [SV17] Yiannis Sakellaridis and Akshay Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque (2017), no. 396, viii+360.
- [SYZ01] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow, *Mirror symmetry is T-duality*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 333–347.
- [Wal85] Jean-Loup Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Mathematica **54** (1985), no. 2, 173–242.
- [Yun16] Zhiwei Yun, *Lectures on Springer theories and orbital integrals*, arXiv preprint arXiv:1602.01451 (2016).
- [YZ17] Zhiwei Yun and Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, Ann. of Math. (2) **186** (2017), no. 3, 767–911.
- [YZ19] ———, *Shtukas and the Taylor expansion of L-functions (II)*, Ann. of Math. (2) **189** (2019), no. 2, 393–526.