

In this note, we summarize the author’s joint work with Xinchun Ma and Pablo Boixeda-Alvarez relating type A combinatorics and torus link invariants with certain homogeneous affine Springer fibers, and point towards some natural follow up questions.

Fix a pair of positive integers M, N . To (M, N) we associate the following objects, each of a completely different flavor:

1. The (flag) affine Springer fiber $\mathcal{F}l_\gamma$ of slope M/N for $G = \mathrm{GL}_N$.
2. A coherent sheaf $\mathcal{F}_{M/N}$ on the Hilbert scheme of N points on the plane.
3. The (M, N) torus link.
4. Representation theory of the rational Cherednik of slope M/N .

Remarkably, all of these objects are related. At present, the strongest form of this relationship takes the form of an equality of polynomials in q, t , and two infinite sets of polynomials X_i and Y_i , which is symmetric in X and in Y . For each of the examples (1)–(4), we extract such a polynomial by considering:

1. The bigraded Frobenius character of the Borel-Moore homology of the positive part of $\mathcal{F}l_\gamma$ (with the Hikita filtration).
2. The $\mathbb{C}^* \times \mathbb{C}^*$ equivariant K theory class of $\mathcal{F}_{M/N}$.
3. The symmetric function lift, constructed in [Wil23], of the KhR link homology [KR08]. Note the KhR homology can be specialized to the HOMFLY-PT link invariant [FYH⁺85, PT88] or to the Alexander or Jones polynomial of the link.
4. Bigraded character of representation of rational Cherednik algebras of minimal support.

Conjecture 1. All four of the invariants above are equal.

Conjecture 1 has a long history of past work: The Borel-Moore homology of the affine Springer fibers were computed in [Hik14] for the coprime case, and in [CM25] in the case $M = kN$ is an integer multiple of N . In [OY16], the representations (4) are explicitly constructed on the equivariant compactly supported cohomology of affine Springer fibers in the coprime case, and in [BAL24] the same is done for Borel-Moore homology in the case when M/N is an integer. The symmetric function lift of the torus link invariants were computed in [Wil23] using the recursive formulas for KhR knot homology in [HM19, GMV20]. Some hard combinatorics, which goes by the name of the “Shuffle Theorem” proved by Carlsson and Mellit [CM18], proves that the computations on affine Springer fibers (1) in the coprime and integral cases line up with the invariants of the torus link invariants (3). On the other hand, in her exceptional thesis, X. Ma constructed the sheaf $\mathcal{F}_{M,N}$ for coprime M and N using mirabolic D modules, computed the polynomial (2), and proved its relation to the representation theory of the rational Cherednik algebra in (4), computing the proof that all of (1) through (4) are equal in this case [Ma24].

In my joint work with X. Ma and P. Boixeda-Alvarez, we prove the non-coprime case of Conjecture 1, completing the proof of the following.

Theorem 2. ([AHM26]) *Conjecture 1 is true for all M, N .*

Our method is to explicitly compute (1) and (2) in the non-coprime case. The computation of (2) (and the construction of such a sheaf $\mathcal{F}_{M/N}$) relies on extensions of the previous work of X. Ma [Ma24]. In this statement, we focus on the computation of (1). To relate (1) and (2), we use a new combinatorial Shuffle theorem for the non-coprime case (known as the “Wilson conjecture” in the literature) which was proved without the Y variables in [KO26b] and is to appear in full generality in [KO26a].

(Flag) Affine Springer fibers and cyclic combinatorics

Throughout this section, $G = \mathrm{GL}_N$. Let $F = \mathbb{C}((\varpi))$ be the field of Laurent series over $\mathbb{C}$ and $\mathcal{O} = \mathbb{C}[[\varpi]]$ its ring of integers. We let $I \subset G(\mathcal{O})$ denote the standard Iwahori subgroup; that is, I consists of all matrices with entries in $\mathcal{O}$ whose reduction mod ϖ is upper triangular. We have an action map

$$a : G_F \times^I \mathfrak{g}_{\mathcal{O}} \rightarrow \mathfrak{g}_F, \quad (g, x) \mapsto gxg^{-1}.$$

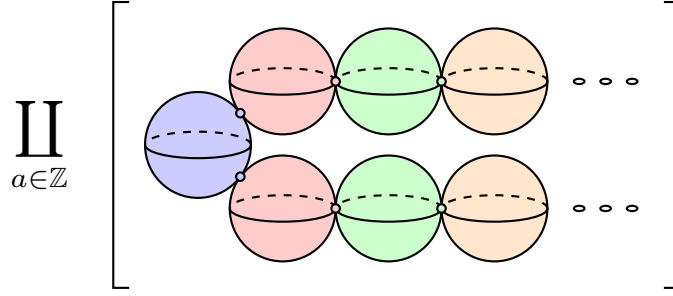


Figure 1: The affine Springer fiber for GL_2 , $\gamma = \mathrm{diag}(t, -t)$. Color coding indicates the Hessenberg stratification in this case.

The affine Springer fiber at a point $\gamma \in \mathfrak{g}(\mathcal{O})$ is the preimage $a^{-1}(\gamma)$. For example, in the case $N = 2$ and $\gamma = \mathrm{diag}(t, -t)$, the corresponding affine Springer fiber is $\mathbb{Z}$ many copies of an infinite union of projective lines (see Figure 1).

It was shown in [KL88] that $\mathcal{F}l_\gamma$ is an equidimensional, connected projective variety locally of finite type whenever γ is regular semisimple over F and topologically nilpotent. We study the particular case when γ is an element with characteristic polynomial $\lambda^N - \varpi^M$. We note that the plane curve singularity $\lambda^N - \varpi^M$ is directly connected to the (M, N) torus link: The intersection of a small ball centered at the singularity with the graph of $\lambda^N - \varpi^M$ is a cone over this torus link (see [Ham25] for some visualizations).

The affine Springer fiber $\mathcal{F}l_\gamma$ depends only on the characteristic polynomial of γ , but with a careful choice of γ , we can stratify $\mathcal{F}l_\gamma$ by parahoric orbits. Namely, let $\mathfrak{P} \subset G(\mathcal{O})$ denote the subgroup of matrices whose reduction mod ϖ lies in standard parabolic of type d^n , where $d = \mathrm{GCD}(M, N)$ and $n = N/d$. Then, with a special choice of γ (modeling the choice made in [Hik14]), it was proven in far greater generality in [GKM06] that the subvarieties $\mathcal{F}l_\gamma \cap \mathfrak{P}wI/I$ are affine space bundles over generalized Hessenberg varieties.¹ When one replaces $\mathfrak{P}$ with an Iwahori I , this becomes a paving by affine spaces, which was considered for various equivalent affine Springer fibers in [Hik14, LS91, GMO26].

However, in this special case, a very special combinatorics can be used to describe these cells, which was developed in [Wil23, GMV17]. We say a set $\Delta \subset \mathbb{Z}$ is an M, N invariant set if it satisfies

- If $r \in \Delta$ then $r + M, r + N \in \Delta$;
- Δ is bounded from below;
- Let a be the minimal element in Δ . Then $\#(\mathbb{Z}_{\geq a} \setminus \Delta) < \infty$.

We will often picture an invariant set as a tuple of paths by labeling $d = \mathrm{GCD}(M, N)$ sheets as in Figure 2, with rows increasing to the left by N and columns increasing upwards by M and drawing lines so that Δ is the set of integers to the left of all paths. A *labeling* of Δ is a function f on the rightmost elements in Δ in each row such that $f(g) \leq f(g + M)$.

Proposition 3. *The nonempty cells $\mathcal{F}l_\gamma \cap \mathfrak{P}wI/I$ are in natural bijection with labeled invariant sets (Δ, f) such that the image of f is exactly $\{1, \dots, N\}$.*

We say that the *positive part* $\mathcal{F}l_\gamma^+$ of $\mathcal{F}l_\gamma$ is the union of all cells whose label (Δ, f) satisfies $\Delta \subset \mathbb{Z}_{>0}$. We have statistics *area* and *lcodinv* associated to (Δ, f) then: $\mathrm{area}(\Delta) = \#(\mathbb{Z}_{>0} \setminus \Delta)$ is the size of the complement of Δ , and $\mathrm{lcodinv}(\Delta, f)$ is given by a more complicated, but entirely elementary combinatorial expression defined in [Wil23]. We prove that when $\mathcal{F}l_\gamma \cap \mathfrak{P}wI/I$ is nonempty, its dimension is $\mathrm{lcodinv}(\Delta, f)$. The area statistic determines a filtration on $\mathcal{F}l_\gamma^+$ which we call the *Hikita filtration*. We expect this to be related, but not exactly equal to, the perverse filtration on the affine Springer fiber; see [KT25, Conjecture 8.10]. The above reduces the computation of Borel-Moore homology to the analogous computation on Hessenberg varieties.

¹The motivation of [GKM06] for studying these pavings was to prove purity of the cohomology of affine Springer fibers. Purity implies among other things the Fundamental Lemma, which was eventually proved via other methods in [Ngo10].

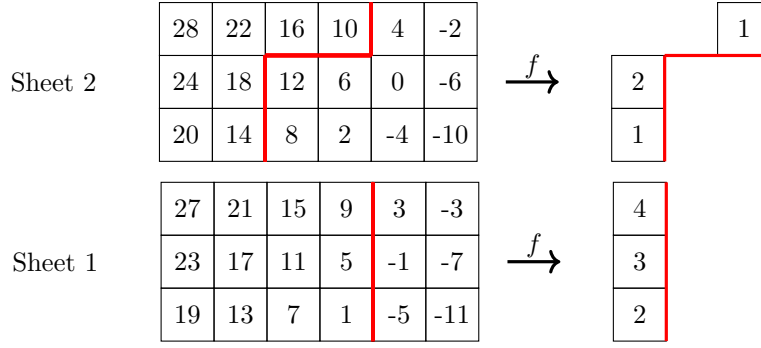


Figure 2: An example of an invariant set Δ for $M = 4$ and $N = 6$, with a labeling f .

Before moving to discuss Hessenberg varieties, we wish to emphasize a crucial difference between the noncoprime and coprime cases: By [OY16], we can put affine Springer fibers for the same M and N in a *family*. We compute that the monodromy action of this family factors through an action of the symmetric group S_d for $d = \text{GCD}(M, N)$. The nontriviality of this action is the essential reason for the nontriviality of the Hessenberg varieties in the paving above.

Generalized Hessenberg varieties and cyclic chromatic functions

A Hessenberg variety very roughly replaces the affine flag variety $\mathcal{Fl} = G(F)/I$ with the usual flag variety G/B . More precisely, a generalized Hessenberg variety for a reductive group H with representation V is a variety

$$\mathcal{H}(x, \mathfrak{m}) = \{g \in H/B \mid g^{-1} \cdot x \in \mathfrak{m}\},$$

where here we can take any $x \in V$ and any B stable subspace $\mathfrak{m} \subset V$, which we call a *Hessenberg space*. Hessenberg varieties were studied for the adjoint representation of G with x regular in $\mathfrak{g}$ (such cases are called *regular Hessenberg varieties*) in [Tym07, Pre13, BC18, BC24] and some examples of generalized Hessenberg varieties for symmetric spaces were considered in [CVX20].

We consider the Hessenberg varieties associated to the group $H = \text{GL}_d^n$, thought of as a block diagonal Levi in $G = \text{GL}_N$. The representation of H is obtained as follows: Let θ denote the automorphism of G obtained by conjugating by $\text{diag}(I_d, \zeta_n I_d, \dots, \zeta_n^{n-1} I_d)$, where ζ_n is a primitive n -th root of unity. We let $V \subset \mathfrak{g}$ be the eigenspace corresponding to eigenvalue ζ_n^b , where bd is the residue of M modulo N . The representations associated to such automorphisms and their invariant theory was considered in [Vin76]. In this case $n = N/d = 2$, θ is the order two involution corresponding to the symmetric space $\text{GL}_{2d}/\text{GL}_d \times \text{GL}_d$.

In the case of regular Hessenberg varieties in type A , the Borel-Moore homology is computed by a fundamental combinatorial function: In this case, the Hessenberg space $\mathfrak{m} \subset \mathfrak{g}$ is determined by a sequence $1 \leq m_1 \leq m_2 \leq \dots \leq m_n = n$ with $m_i \geq i$, which in turn determines a partially ordered set on $\{1, \dots, N\}$ by the condition that $i < j$ if and only if $j \geq m_i$. This is a special type of POSET called a *unit interval order*. We are interested in the *chromatic symmetric function* of the incomparability graph G . This is the polynomial in t and variables X_i which sums over all *proper colorings* κ of G , measuring certain invariants called ascents of the coloring. While these functions are generally only quasisymmetric for a general graph G , it turns out that they are symmetric (in X variables) for unit interval orders [SW16, Corollary 4.6]. It was conjectured in [SW12] and proven in [BC18] that the Frobenius character of the regular Hessenberg variety agrees with the chromatic symmetric function of the incomparability graph associated to the corresponding Hessenberg space.

In our case, computing the Frobenius character of our generalized Hessenberg varieties produces a completely combinatorially defined function which is natural generalization of the chromatic symmetric function. Our function is a generalization of the digraph definition of the chromatic symmetric function (see [BC18]), and it works by replacing a digraph on N vertices by a *cyclic partite* digraph, with n groups of d vertices. Our function is a double sum: First summing over possible choices of cycles, which collapse the

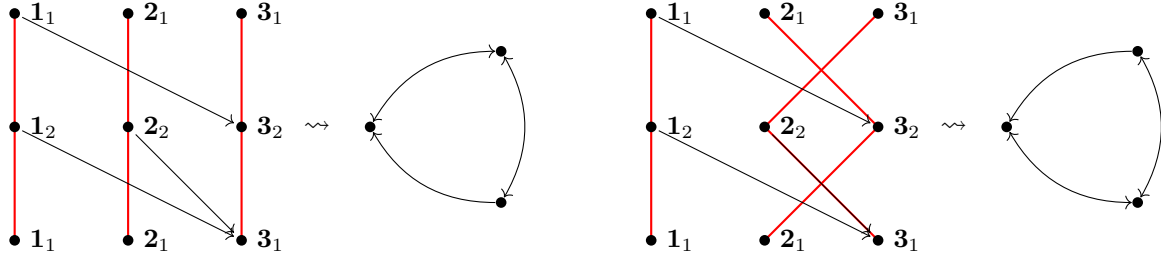


Figure 3: Possible collapsing by cycles of cyclic 2-partite digraphs ($n = 2$, $d = 3$). The black arrows denote the arrows in the complement digraph. The red lines indicate possible choices of disjoint cycles in D . The graphs drawn are the quotient digraphs D/L .

cyclic, partite digraph to a usual digraph on d vertices. The second sum is over the same set as classical chromatic functions for the resulting digraph on d vertices. See Figure 3. We call this new Hessenberg variety $\Xi_{\mathbf{m}}(X, t)$.

Theorem 4. *The S_d Frobenius character of the Borel-Moore homology of the generalized Hessenberg variety $\mathcal{H}(x, \mathbf{m})$ for x regular, semisimple is the cyclic chromatic function $\Xi_{\mathbf{m}}(X, t)$.*

In particular, $\Xi_{\mathbf{m}}(X, t)$ is a symmetric function in X and is palindromic in t .

To prove Theorem 4, we use similar methods to [Pre13, BC18]. Namely, we show that the intersection of the Hessenberg variety above with the Schubert cells BwB/B in the flag variety G/B is affine. This affine paving allows us to compute the Betti numbers of generalized Hessenberg varieties for all regular x . We complete the proof by assembling a family of Hessenberg spaces, interpreting the S_d action as monodromy of the family, and applying the local invariant cycles theorem of Beilinson, Bernstein, and Deligne [BBDG18] to relate the Frobenius character of regular, semisimple elements to the Poincaré polynomials of regular, but not necessarily semisimple, elements.

Future directions

Conjecture 1 and Theorem 2 are restricted to the case of torus links. There are many open questions about generalizing to links beyond this case. For example, in [KT23], using methods imported from Waldspurger’s computations of Shalika germs, it was shown that point counts on affine Springer fibers (called “orbital integrals”) follow the same recursions as cablings for knot invariants.

Project 5. Extend Theorem 2 beyond the torus link case for *elliptic* $\gamma \in \mathfrak{g}(\mathcal{O})$.

To do so, we wish to apply global methods. That is, we wish to consider moduli spaces of Higgs bundles, and consider the affine Springer fibers in pseudo-family via the product formula. As the product formula is a statement about the affine Springer fiber modulo stabilizers, this perspective most easily recovers computations of orbital integrals, or point counts on the affine Springer fiber.

Kivinen’s computations give combinatorial formulas for the point counts on affine Springer fibers [KT23], which in turn determines the Euler characteristic of the compactly supported cohomology. Assuming purity, this directly gives information about the Borel-Moore homology. Unfortunately, purity of affine Springer fibers, even in type A , remains a long standing conjecture. But at least in the *equivalued* case, where [GKM06] implies purity, there is hope to compute Borel-Moore homology via global methods.

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